例題と演習で学ぶ　微分積分学　演習問題解答
(第 6 刷にも対応)

第 2 章

2.1. C を積分定数とする．

(1) $\displaystyle\int x^4 dx = \frac{1}{5}x^5 + C$

(2) $\displaystyle\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1} + C = -\frac{1}{x} + C$

(3) $\displaystyle\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$

(4) $\displaystyle\int (x+1)^3 dx = \int (x^3 + 3x^2 + 3x + 1)dx = \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

(5) $\displaystyle\int \frac{1}{9+x^2}dx = \int \frac{1}{3^2+x^2}dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

(6) $\displaystyle\int \frac{\cos x}{\sin x}dx = \int \frac{(\sin x)'}{\sin x}dx = \log|\sin x| + C$

(7) $\displaystyle\int \frac{x+1}{x^2+2x+3}dx = \frac{1}{2}\int \frac{(x^2+2x+3)'}{x^2+2x+3}dx = \frac{1}{2}\log(x^2+2x+3) + C$

(8) $\displaystyle\int \frac{3x^2 - 4\sqrt[3]{x}}{x\sqrt{x}}dx = \int \left(3x^{\frac{1}{2}} - 4x^{-\frac{7}{6}}\right)dx = 2x^{\frac{3}{2}} + 24x^{-\frac{1}{6}} + C$

(9)
$$\begin{aligned}\int x^2\left(x - \frac{2}{x}\right)^3 dx &= \int x^2\left(x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}\right)dx \\ &= \int \left(x^5 - 6x^3 + 12x - \frac{8}{x}\right)dx \\ &= \frac{1}{6}x^6 - \frac{3}{2}x^4 + 6x^2 - 8\log|x| + C\end{aligned}$$

(10) $\displaystyle\int \frac{2x^3 - 3\sqrt{x}}{x^2\sqrt{x}}dx = \int \left(2x^{\frac{1}{2}} - 3x^{-2}\right)dx = \frac{4}{3}x^{\frac{3}{2}} + 3x^{-1} + C$

(11)
$$\begin{aligned}\int \frac{(\sqrt{x}-1)^2}{2x\sqrt{x}}dx = \int \frac{x - 2\sqrt{x} + 1}{2x\sqrt{x}}dx &= \int \left(\frac{1}{2}x^{-\frac{1}{2}} - x^{-1} + \frac{1}{2}x^{-\frac{3}{2}}\right)dx \\ &= x^{\frac{1}{2}} - \log|x| - x^{-\frac{1}{2}} + C\end{aligned}$$

(12) $\displaystyle\int x\sqrt{x}dx = \int x^{\frac{3}{2}}dx = \frac{2}{5}x^{\frac{5}{2}} + C$

(13) $\displaystyle\int \frac{x}{\sqrt[3]{x}}dx = \int x^{\frac{2}{3}}dx = \frac{3}{5}x^{\frac{5}{3}} + C$

(14) $\displaystyle\int \frac{2x+1}{x^2}dx = \int(2x^{-1}+x^{-2})dx = 2\log|x| - x^{-1} + C$

(15) $\displaystyle\int \frac{3x-4}{\sqrt{x}}dx = \int\left(3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right)dx = 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + C$

(16) $$\begin{aligned}\int \frac{(\sqrt{x}-1)^3}{\sqrt{x}}dx &= \int \frac{x\sqrt{x}-3x+3\sqrt{x}-1}{\sqrt{x}}dx \\ &= \int\left(x - 3x^{\frac{1}{2}} + 3 - x^{-\frac{1}{2}}\right)dx \\ &= \frac{1}{2}x^2 - 2x^{\frac{3}{2}} + 3x - 2x^{\frac{1}{2}} + C\end{aligned}$$

(17) $\displaystyle\int(3\sin x - 4\cos x)dx = -3\cos x - 4\sin x + C$

(18) $\displaystyle\int \frac{1-\cos^3 x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x} - \cos x\right)dx = \tan x - \sin x + C$

(19) $\displaystyle\int(e^x - 2^x)dx = e^x - \frac{2^x}{\log 2} + C$

(20) $\displaystyle\int\left(x^{\frac{1}{4}} - x^{\frac{1}{3}}\right)^2 dx = \int\left(x^{\frac{1}{2}} - 2x^{\frac{7}{12}} + x^{\frac{2}{3}}\right)dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{24}{19}x^{\frac{19}{12}} + \frac{3}{5}x^{\frac{5}{3}} + C$

2.2. C を積分定数とする.

(1) $$\begin{aligned}\int x\sin x dx &= \int x(-\cos x)'dx = -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

(2) $\displaystyle\int xe^{-x}dx = \int x(-e^{-x})'dx = -xe^{-x} + \int e^{-x}dx = -(x+1)e^{-x} + C$

(3) $$\begin{aligned}\int(x+3)\cos 2x dx &= \int(x+3)\left(\frac{1}{2}\sin 2x\right)'dx \\ &= \frac{x+3}{2}\sin 2x - \frac{1}{2}\int \sin 2x dx \\ &= \frac{x+3}{2}\sin 2x + \frac{1}{4}\cos 2x + C\end{aligned}$$

(4) $$\begin{aligned}\int x\log x dx &= \int\left(\frac{1}{2}x^2\right)'\log x dx \\ &= \frac{1}{2}x^2\log x - \frac{1}{2}\int x dx = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C\end{aligned}$$

(5) $$\begin{aligned}\int \log(x+1)dx &= \int(x+1)'\log(x+1)dx \\ &= (x+1)\log(x+1) - \int dx = (x+1)\log(x+1) - x + C\end{aligned}$$

(6) $\displaystyle\int (x-1)e^x dx = (x-1)e^x - \int e^x dx = (x-2)e^x + C$

(7) $\displaystyle\int \sqrt{x}\log x dx = \frac{2}{3}x^{\frac{3}{2}}\log x - \frac{2}{3}\int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}\log x - \frac{4}{9}x^{\frac{3}{2}} + C$

(8) $$\begin{aligned}\int (1-2x)\sin x dx &= (1-2x)(-\cos x) - 2\int \cos x dx \\ &= (2x-1)\cos x - 2\sin x + C\end{aligned}$$

(9) $\displaystyle\int xa^x dx = x\frac{a^x}{\log a} - \frac{1}{\log a}\int a^x dx = x\frac{a^x}{\log a} - \frac{a^x}{(\log a)^2} + C$

(10) $\displaystyle I = \int e^x \cos x dx$ とおく.

$$\begin{aligned}I = \int e^x \cos x dx &= e^x \cos x + \int e^x \sin x dx \\ &= e^x \cos x + \left(e^x \sin x - \int e^x \cos x dx\right) \\ &= e^x(\sin x + \cos x) - I\end{aligned}$$

よって, $\displaystyle\int e^x \cos x dx = \frac{e^x}{2}(\sin x + \cos x) + C.$

(11) $$\begin{aligned}\int \sin^3 x dx &= \int \sin x \cdot \sin^2 x dx \\ &= -\cos x \sin^2 x + \int \cos x \cdot 2\sin x \cos x dx \\ &= -\cos x \sin^2 x + 2\int \sin x(1-\sin^2 x)dx \\ &= -\cos x \sin^2 x - 2\cos x - 2\int \sin^3 x dx\end{aligned}$$

よって, $\displaystyle\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$

(12) $$\begin{aligned}\int x\log(x^2+1)dx &= \int \left(\frac{x^2+1}{2}\right)'\log(x^2+1)dx \\ &= \frac{x^2+1}{2}\log(x^2+1) - \frac{1}{2}\int 2x dx \\ &= \frac{x^2+1}{2}\log(x^2+1) - \frac{1}{2}x^2 + C\end{aligned}$$

(13) $I = \displaystyle\int \sin^5 x dx$ とおく.

$$\begin{aligned} I = \int \sin^5 x dx &= \int \sin x \sin^4 x dx \\ &= -\cos x \sin^4 x + \int \cos x \cdot 4\sin^3 x \cos x dx \\ &= -\cos x \sin^4 x + 4\int (1-\sin^2 x)\sin^3 x dx \\ &= -\cos x \sin^4 x + 4\int \sin^3 x dx - 4I \end{aligned}$$

よって,

$$\begin{aligned} \int \sin^5 x dx &= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\int \sin^3 x dx \\ &= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x\right) + C \\ &= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C \end{aligned}$$

(14) $\displaystyle\int xe^{x+1}dx = xe^{x+1} - \int e^{x+1}dx = (x-1)e^{x+1} + C$

(15)
$$\begin{aligned} \int x\log(x+2)dx &= \int \left(\frac{x^2-4}{2}\right)' \log(x+2)dx \\ &= \frac{x^2-4}{2}\log(x+2) - \frac{1}{2}\int (x^2-4)\cdot\frac{1}{x+2}dx \\ &= \frac{x^2-4}{2}\log(x+2) - \frac{1}{2}\int (x-2)dx \\ &= \frac{x^2-4}{2}\log(x+2) - \frac{1}{4}(x^2-4x) + C \end{aligned}$$

(16)
$$\begin{aligned} \int e^x\log(e^x+1)dx &= \int (e^x+1)'\log(e^x+1)dx \\ &= (e^x+1)\log(e^x+1)dx - \int (e^x+1)\cdot\frac{e^x}{e^x+1}dx \\ &= (e^x+1)\log(e^x+1)dx - \int e^x dx \\ &= (e^x+1)\log(e^x+1)dx - e^x + C \end{aligned}$$

(17)
$$\begin{aligned} \int x\sin(x-1)dx &= -x\cos(x-1) + \int \cos(x-1)dx \\ &= -x\cos(x-1) + \sin(x-1) \end{aligned}$$

(18) $\displaystyle\int x^5 \log x dx = \frac{1}{6}x^6 \log x - \frac{1}{6}\int x^5 dx = \frac{1}{6}x^6 \log x - \frac{1}{36}x^6 + C$

(19) $\displaystyle I = \int \cos^3 x dx$ とおく.

$$\begin{aligned} I = \int \cos x \cdot \cos^2 x dx &= \sin x \cos^2 x - \int \sin x \cdot 2\cos x(-\sin x) dx \\ &= \sin x \cos^2 x + 2\int \cos x dx - 2\int \cos^3 x dx \\ &= \sin x \cos^2 x + 2\sin x - 2I \end{aligned}$$

よって, $\displaystyle\int \cos^3 x dx = \frac{1}{3}\sin x \cos^2 x + \frac{2}{3}\sin x + C.$

(20) 部分積分の計算を 3 回行えばよい.

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + \int 3x^2 \cos x dx \\ &= -x^3 \cos x + 3\left(x^2 \sin x - 2\int x \sin x dx\right) \\ &= -x^3 \cos x + 3x^2 \sin x - 6\left(-x\cos x + \int \cos x dx\right) \\ &= -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C \end{aligned}$$

2.3.

(1) $\log x = t$ とおくと, $\displaystyle\frac{1}{x}dx = dt$. よって,

$$\int \frac{1}{x \log x} dx = \int \frac{1}{t} dt = \log|t| + C = \log|\log x| + C$$

(2) $\displaystyle\int \frac{1}{x^2+2x+2} dx = \int \frac{1}{(x+1)^2+1} dx$. ここで, $x+1=t$ とおくと, $dx = dt$. よって,

$$\int \frac{1}{x^2+2x+2} dx = \int \frac{1}{t^2+1} dt = \tan^{-1} t + C = \tan^{-1}(x+1) + C$$

(3) $\displaystyle\int \frac{2x}{x^2-2x+5}dx = \int \frac{2x}{(x-1)^2+4}dx$. ここで, $x-1=t$ とおくと, $dx = dt$. よって,

$$\begin{aligned}\int \frac{2x}{x^2-2x+5}dx &= \int \frac{2t+2}{t^2+4}dt \\ &= \int \frac{2t}{t^2+4}dt + \int \frac{2}{t^2+4}dt \\ &= \log(t^2+4) + \tan^{-1}\frac{t}{2} + C \\ &= \log(x^2-2x+5) + \tan^{-1}\frac{x-1}{2} + C\end{aligned}$$

(4) $2x = t$ とおくと, $2dx = dt$, すなわち $dx = \dfrac{1}{2}dt$. よって,

$$\int \cos 2x dx = \frac{1}{2}\int \cos t dt = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C$$

(5) $\displaystyle\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx$. ここで, $2x = t$ とおくと, $2dx = dt$, すなわち $dx = \dfrac{1}{2}dt$. よって,

$$\int \sin^2 x dx = \frac{1}{2}\int \frac{1-\cos t}{2}dt = \frac{1}{4}t - \frac{1}{4}\sin t + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(6) $\displaystyle\int \cos^2 x dx = \int \frac{1+\cos 2x}{2}dx$. ここで, $2x = t$ とおくと, $2dx = dt$, すなわち $dx = \dfrac{1}{2}dt$. よって,

$$\int \cos^2 x dx = \frac{1}{2}\int \frac{1+\cos t}{2}dt = \frac{1}{4}t + \frac{1}{4}\sin t + C = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

(7) $3x = t$ とおくと, $3dx = dt$, すなわち, $dx = \dfrac{1}{3}dt$. よって,

$$\int e^{3x}dx = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + C = \frac{1}{3}e^{3x} + C$$

(8) $\sin x = t$ とおくと, $\cos x dx = dt$. よって,

$$\int \cos x e^{\sin x}dx = \int e^t dt = e^t + C = e^{\sin x} + C$$

(9) $3x = t$ とおくと, $3dx = dt$, すなわち, $dx = \dfrac{1}{3}dt$. よって,

$$\int \frac{1}{\sqrt{4-9x^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{4-t^2}}dt = \frac{1}{3}\sin^{-1}\frac{t}{2} + C = \frac{1}{3}\sin^{-1}\frac{3}{2}x + C$$

(10) $\log(1+x^2) = t$ とおくと, $\dfrac{2x}{1+x^2}dx = dt$, すなわち, $\dfrac{x}{1+x^2}dx = \dfrac{1}{2}dt$. よって,

$$\int \frac{x\log(1+x^2)}{1+x^2}dx = \frac{1}{2}\int tdt = \frac{1}{4}t^2 + C = \frac{1}{4}\left\{\log(1+x^2)\right\}^2 + C$$

(11) $1 - x^2 = t$ とおくと, $-2xdx = dt$, すなわち, $xdx = -\dfrac{1}{2}dt$. よって,

$$\int x\sqrt{1-x^2}dx = -\frac{1}{2}\int \sqrt{t}dt = -\frac{1}{3}t^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

(12) $3 - x = t$ とおくと, $-dx = dt$, すなわち $dx = -dt$. よって,

$$\begin{aligned}\int \frac{x^2}{(3-x)^2}dx &= -\int \frac{(3-t)^2}{t^2}dt \\ &= -\int \frac{t^2-6t+9}{t^2}dt \\ &= -\int \left(1 - \frac{6}{t} + \frac{9}{t^2}\right)dt \\ &= -t + 6\log|t| + \frac{9}{t} + C = x - 3 + 6\log|x-3| + \frac{9}{x-3} + C\end{aligned}$$

(13) $\displaystyle\int \frac{e^x}{1+e^x}dx = \int \frac{(1+e^x)'}{1+e^x}dx = \log(1+e^x) + C$

(14) $3x + 2 = t$ とおくと, $3dx = dt$, すなわち, $dx = \dfrac{1}{3}dt$. よって,

$$\int (3x+2)^4dx = \frac{1}{3}\int t^4dt = \frac{1}{15}t^5 + C = \frac{1}{15}(3x+2)^5 + C$$

(15) $2 - x = t$ とおくと, $-dx = dt$, すなわち, $dx = -dt$. よって,

$$\int \sqrt{2-x}dx = -\int \sqrt{t}dt = -\frac{2}{3}(2-x)^{\frac{3}{2}} + C$$

(16) $\displaystyle\int \frac{1}{4x+1}dx = \frac{1}{4}\int \frac{(4x+1)'}{4x+1}dx = \frac{1}{4}\log|4x+1| + C$

(17) $5x + 2 = t$ とおくと, $5dx = dt$, すなわち, $dx = \dfrac{1}{5}dt$. よって,

$$\int \frac{1}{(5x+2)^3}dx = \frac{1}{5}\int \frac{1}{t^3}dt = -\frac{1}{10t^2} + C = -\frac{1}{10(5x+2)^2} + C$$

(18) $4x + 1 = t$ とおくと, $4dx = dt$, すなわち $dx = \dfrac{1}{4}dt$. よって,

$$\int e^{4x+1}dx = \frac{1}{4}\int e^tdt = \frac{1}{4}e^t + C = \frac{1}{4}e^{4x+1} + C$$

(19) $\dfrac{\pi}{3}-2x=t$ とおくと, $-2dx=dt$, すなわち, $dx=-\dfrac{1}{2}dt$. よって,

$$\int \sin\left(\frac{\pi}{3}-2x\right)dx=-\frac{1}{2}\int \sin t dt=\frac{1}{2}\cos t+C=\frac{1}{2}\cos\left(\frac{\pi}{3}-2x\right)+C$$

(20) $2x-1=t$ とおくと, $x=\dfrac{t+1}{2}$. また $2dx=dt$, すなわち, $dx=\dfrac{1}{2}dt$. よって,

$$\begin{aligned}\int x\sqrt{2x-1}dx &= \frac{1}{2}\int \frac{t+1}{2}\cdot\sqrt{t}dt\\ &= \frac{1}{4}\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)dt\\ &= \frac{1}{10}t^{\frac{5}{2}}+\frac{1}{6}t^{\frac{3}{2}}+C\\ &= \frac{1}{10}(2x-1)^{\frac{5}{2}}+\frac{1}{6}(2x-1)^{\frac{3}{2}}+C\\ &= \frac{1}{30}(2x-1)^{\frac{3}{2}}\{3(2x-1)+5\}+C\\ &= \frac{1}{15}(2x-1)^{\frac{3}{2}}(3x+1)+C\end{aligned}$$

(21) $x+1=t$ とおくと, $x=t-1$. また $dx=dt$. よって,

$$\begin{aligned}\int x\sqrt{x+1}dx &= \int (t-1)\sqrt{t}dt\\ &= \int\left(t^{\frac{3}{2}}-t^{\frac{1}{2}}\right)dt\\ &= \frac{2}{5}t^{\frac{5}{2}}-\frac{2}{3}t^{\frac{3}{2}}+C\\ &= \frac{2}{15}t^{\frac{3}{2}}(3t-5)+C\\ &= \frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2)+C\end{aligned}$$

(22) $3x-1=t$ とおくと, $3dx=dt$, すなわち $dx=\dfrac{1}{3}dt$. よって,

$$\begin{aligned}\int \frac{x}{3x-1}dx &= \frac{1}{3}\int \frac{\frac{1}{3}(t+1)}{t}dt\\ &= \frac{1}{9}\int\left(1+\frac{1}{t}\right)dt\\ &= \frac{1}{9}t+\log|t|+C\\ &= \frac{1}{9}(3x-1)+\frac{1}{9}\log|3x-1|+C\end{aligned}$$

(23) $x-1=t$ とおくと, $dx=dt$. よって,

$$\begin{aligned}\int \frac{x^2}{\sqrt{x-1}}dx &= \int \frac{(t+1)^2}{\sqrt{t}}dt \\ &= \int \left(t^{\frac{3}{2}}+2t^{\frac{1}{2}}+t^{-\frac{1}{2}}\right)dt \\ &= \frac{2}{5}t^{\frac{5}{2}}+\frac{4}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}+C \\ &= \frac{2}{15}t^{\frac{1}{2}}\left(3t^2+10t+15\right)+C \\ &= \frac{2}{15}(3x^2+4x+8)\sqrt{x-1}+C\end{aligned}$$

(24) $x^2+1=t$ とおくと, $2xdx=dt$. よって,

$$\begin{aligned}\int 2x\sqrt{x^2+1}dx &= \int \sqrt{t}dt \\ &= \frac{2}{3}t^{\frac{3}{2}}+C \\ &= \frac{2}{3}(x^2+1)^{\frac{3}{2}}+C\end{aligned}$$

(25) $x^3-1=t$ とおくと, $3x^2dx=dt$, すなわち $x^2dx=\dfrac{1}{3}dt$. よって,

$$\begin{aligned}\int x^2\sqrt{x^3-1}dx &= \frac{1}{3}\int \sqrt{t}dt \\ &= \frac{2}{9}t^{\frac{3}{2}}+C \\ &= \frac{2}{9}(x^3-1)^{\frac{3}{2}}+C\end{aligned}$$

(26) $\sin x=t$ とおくと, $\cos xdx=dt$. よって,

$$\int \sin^3 x\cos xdx = \int t^3dt = \frac{1}{4}t^4+C = \frac{1}{4}\sin^4 x+C$$

(27) $2x-3=t$ とおくと, $2dx=dt$, すなわち $dx=\dfrac{1}{2}dt$. よって,

$$\int \sqrt[3]{(2x-3)^2}dx = \frac{1}{2}\int t^{\frac{2}{3}}dt = \frac{1}{2}\cdot\frac{3}{5}t^{\frac{5}{3}}+C = \frac{3}{10}(2x-3)^{\frac{5}{3}}+C$$

(28) $3x=t$ とおくと, $3dx=dt$, すなわち $dx=\dfrac{1}{3}dt$. よって,

$$\int \frac{1}{\cos^2 3x}dx = \frac{1}{3}\int \frac{1}{\cos^2 t}dt = \frac{1}{3}\tan t+C = \frac{1}{3}\tan 3x+C$$

(29) $2x+3=t$ とおくと, $2dx=dt$, すなわち $dx=\frac{1}{2}dt$. よって,

$$\begin{aligned}\int x(2x+3)^3dx &= \frac{1}{2}\int \frac{t-3}{2}t^3dt \\ &= \frac{1}{4}\int (t^4-3t^3)dt \\ &= \frac{1}{4}\left(\frac{1}{5}t^5-\frac{3}{4}t^4\right)+C \\ &= \frac{1}{80}t^4(4t-15)+C \\ &= \frac{1}{80}(8x-3)(2x+3)^4+C\end{aligned}$$

(30) $2x+1=t$ とおくと, $2dx=dt$, すなわち $dx=\frac{1}{2}dt$. よって,

$$\begin{aligned}\int \frac{2x-1}{\sqrt{2x+1}}dx &= \frac{1}{2}\int \frac{t-2}{\sqrt{t}}dt \\ &= \frac{1}{2}\int (t^{\frac{1}{2}}-2t^{-\frac{1}{2}})dt \\ &= \frac{1}{2}\left(\frac{2}{3}t^{\frac{3}{2}}-4t^{\frac{1}{2}}\right)+C \\ &= \frac{t^{\frac{1}{2}}}{3}(t-6)+C \\ &= \frac{1}{3}(2x-5)\sqrt{2x+1}+C\end{aligned}$$

(31) $4-3x^2=t$ とおくと, $-6xdx=dt$, すなわち $xdx=-\frac{1}{6}dt$. よって,

$$\begin{aligned}\int \frac{x}{\sqrt{4-3x^2}}dx &= -\frac{1}{6}\int \frac{1}{\sqrt{t}}dt \\ &= -\frac{1}{6}\int t^{-\frac{1}{2}}dt \\ &= -\frac{1}{3}t^{\frac{1}{2}}+C \\ &= -\frac{1}{3}(4-3x^2)^{\frac{1}{2}}+C\end{aligned}$$

(32) $3x^3=t$ とおくと, $9x^2dx=dt$, すなわち, $x^2dx=\frac{1}{9}dt$. よって,

$$\int x^2e^{3x^3}dx=\frac{1}{9}\int e^tdt=\frac{1}{9}e^t+C=\frac{1}{9}e^{3x^3}+C$$

2.4.

(1) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち $\sin x dx = -dt$. よって,

$$\int \frac{\sin x}{\cos^2 x}dx = \int \left(-\frac{1}{t^2}\right) dt = \frac{1}{t} + C = \frac{1}{\cos x} + C$$

(2) $x^2 = t$ とおくと, $2xdx = dt$. よって,

$$\int \frac{2x}{x^4+1}dx = \int \frac{1}{t^2+1}dt = \tan^{-1} t + C = \tan^{-1} x^2 + C$$

(3) $\sqrt{e^x - 1} = t$ とおくと, $e^x = t^2 + 1$, $\dfrac{e^x}{2\sqrt{e^x-1}}dx = dt$, すなわち, $dx = \dfrac{2t}{t^2+1}dt$. よって,

$$\begin{aligned}\int \sqrt{e^x-1}dx &= \int t \cdot \frac{2t}{t^2+1}dt \\ &= 2\int \frac{t^2}{t^2+1}dt \\ &= 2\int \left(1 - \frac{1}{t^2+1}\right) dt \\ &= 2t - 2\tan^{-1} t + C \\ &= 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C\end{aligned}$$

(4) $x = \sin t$ とおくと, $dx = \cos t dt$. よって,

$$\begin{aligned}\int \sqrt{1-x^2}dx &= \int \sqrt{1-\sin^2 t}\cos t dt \\ &= \int \cos^2 t dt \\ &= \int \frac{1+\cos 2t}{2}dt \\ &= \frac{1}{2}t + \frac{1}{4}\sin 2t + C \\ &= \frac{1}{2}\sin^{-1} x + \frac{1}{2}\sin(\sin^{-1} x)\cos(\sin^{-1} x) + C \\ &= \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C\end{aligned}$$

(5) $\sin x = t$ とおくと, $\cos x dx = dt$. よって,

$$\int \sin^2 x \cos x dx = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}\sin^3 x + C$$

(6) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち $\sin x dx = -dt$. よって,

$$\int \sqrt{\cos x} \sin x dx = -\int \sqrt{t} dt = -\frac{2}{3} t^{\frac{3}{2}} + C = -\frac{2}{3} \cos^{\frac{3}{2}} x + C$$

2.5.

(1) $\dfrac{1}{(x+1)(x-2)(x+5)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x+5}$ とおく.
両辺を $(x+1)(x-2)(x+5)$ 倍すると,

$$1 = A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)$$

この式に $x = -1$ を代入すると, $A = -\dfrac{1}{12}$, $x = 2$ を代入すると, $B = \dfrac{1}{21}$, $x = -5$ を代入すると, $C = \dfrac{1}{28}$. よって,

$$\frac{1}{(x+1)(x-2)(x+5)} = -\frac{1}{12(x+1)} + \frac{1}{21(x-2)} + \frac{1}{28(x+5)}$$

(2) $\dfrac{3x-2}{x(x-1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$ とおく.
両辺を $x(x-1)(x-2)$ 倍すると,

$$3x - 2 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1)$$

この式に $x = 0$ を代入すると, $A = -1$, $x = 1$ を代入すると, $B = -1$, $x = 2$ を代入すると, $C = 2$. よって,

$$\frac{3x-2}{x(x-1)(x-2)} = -\frac{1}{x} - \frac{1}{x-1} + \frac{2}{x-2}$$

(3) $\dfrac{2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $(x-1)(x^2+1)$ 倍すると,

$$2x = A(x^2+1) + (Bx+C)(x-1)$$

この式に $x = 1$ を代入すると, $A = 1$, $x = 0$ を代入すると, $0 = A - C$ より $C = 1$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = -1$. よって,

$$\frac{2x}{(x-1)(x^2+1)} = \frac{1}{x-1} - \frac{x-1}{x^2+1}$$

(4) $\dfrac{2x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $x(x^2+1)$ 倍すると,

$$\begin{aligned} 2x + 1 &= A(x^2+1) + (Bx+C)x \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

これより, $A = 1$, $C = 2$, $B = -1$. よって,

$$\frac{2x+1}{x(x^2+1)} = \frac{1}{x} - \frac{x-2}{x^2+1}$$

(5) $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $(x+1)(x^2+1)$ 倍すると,

$$1 = A(x^2+1) + (Bx+C)(x+1)$$

この式に $x = -1$ を代入すると, $A = \dfrac{1}{2}$, $x = 0$ を代入すると, $1 = A + C$ より $C = \dfrac{1}{2}$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = -\dfrac{1}{2}$.
よって,

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)}$$

(6) $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $x(x^2+1)$ 倍すると,

$$\begin{aligned} x+1 &= A(x^2+1) + (Bx+C)x \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

これより, $A = 1$, $C = 1$, $B = -1$. よって,

$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} - \frac{x-1}{x^2+1}$$

(7) $\dfrac{2x-1}{(x+2)(x^2+1)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $(x+2)(x^2+1)$ 倍すると,

$$2x-1 = A(x^2+1) + (Bx+C)(x+2)$$

この式に $x = -2$ を代入すると, $A = -1$, $x = 0$ を代入すると, $-1 = A + 2C$ より $C = 0$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = 1$. よって,

$$\frac{2x-1}{(x+2)(x^2+1)} = -\frac{1}{x+2} + \frac{x}{x^2+1}$$

(8) $\dfrac{5x+1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ とおく.
両辺を $(x+1)(x^2+1)$ 倍すると,

$$5x+1 = A(x^2+1) + (Bx+C)(x+1)$$

この式に $x = -1$ を代入すると, $A = -2$, $x = 0$ を代入すると, $1 = A + C$ より $C = 3$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = 2$. よって,

$$\frac{5x+1}{(x+1)(x^2+1)} = -\frac{2}{x+1} + \frac{2x+3}{x^2+1}$$

(9) $\dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ とおく.
両辺を $x(x+1)^2$ 倍すると,

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

この式に $x = 0$ を代入すると, $A = 1$, $x = -1$ を代入すると, $C = -1$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = -1$. よって,

$$\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

(10) $\dfrac{x}{(x-1)(x-2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$ とおく.
両辺を $(x-1)(x-2)^2$ 倍すると,

$$x = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$$

この式に $x = 1$ を代入すると, $A = 1$, $x = 2$ を代入すると, $C = 2$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = -1$. よって,

$$\frac{x}{(x-1)(x-2)^2} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

(11) $\dfrac{x-7}{(x+2)(x-1)^2} = \dfrac{A}{x+2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$ とおく.
両辺を $(x+2)(x-1)^2$ 倍すると,

$$x - 7 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

この式に $x = 1$ を代入すると, $C = -2$, $x = -2$ を代入すると, $A = -1$. 両辺の x^2 の係数を比較すると, $0 = A + B$ より $B = 1$. よって,

$$\frac{x-7}{(x+2)(x-1)^2} = -\frac{1}{x+2} + \frac{1}{x-1} - \frac{2}{(x-1)^2}$$

(12) $\dfrac{x+2}{(x^2-1)^2} = \dfrac{x+2}{(x-1)^2(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$
とおく. 両辺を $(x^2-1)^2$ 倍すると,

$$x + 2 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

この式に $x = 1$ を代入すると, $B = \dfrac{3}{4}$, $x = -1$ を代入すると, $D = \dfrac{1}{4}$, $x = 0$ を代入すると, $-A + B + C + D = 2$ より, $-A + C = 1$. 両辺の x^3 の係数を比較すると, $0 = A + C$. これより, $A = -\dfrac{1}{2}$, $C = \dfrac{1}{2}$. よって,

$$\frac{x+2}{(x^2-1)^2} = -\frac{1}{2(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x+1)} + \frac{1}{4(x+1)^2}$$

(13) $\dfrac{x^2}{x^4-1}=\dfrac{x^2}{(x^2-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}$ とおく.
両辺を x^4-1 倍すると,

$$x^2=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

この式に $x=1$ を代入すると, $A=\dfrac{1}{4}$, $x=-1$ を代入すると, $B=-\dfrac{1}{4}$, $x=0$ を代入すると, $A-B-D=0$ より, $D=\dfrac{1}{2}$. 両辺の x^3 の係数を比較すると, $A+B+C=0$ より, $C=0$. よって,

$$\frac{x^2}{x^4-1}=\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(x^2+1)}$$

(14) $\dfrac{1}{x^4-1}=\dfrac{1}{(x^2-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}$ とおく.
両辺を x^4-1 倍すると,

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

この式に $x=1$ を代入すると, $A=\dfrac{1}{4}$, $x=-1$ を代入すると, $B=-\dfrac{1}{4}$, $x=0$ を代入すると, $A-B-D=1$ より, $D=-\dfrac{1}{2}$. 両辺の x^3 の係数を比較すると, $A+B+C=0$ より, $C=0$. よって,

$$\frac{1}{x^4-1}=\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$$

(15) $\dfrac{1}{x^3-1}=\dfrac{1}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$ とおく.
両辺を x^3-1 倍すると,

$$1=A(x^2+x+1)+(Bx+C)(x-1)$$

この式に $x=1$ を代入すると, $A=\dfrac{1}{3}$, $x=0$ を代入すると, $A-C=1$ より, $C=-\dfrac{2}{3}$. 両辺の x^2 の係数を比較すると, $A+B=0$ より, $B=-\dfrac{1}{3}$. よって,

$$\frac{1}{x^3-1}=\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$$

(16) $\dfrac{3x^3+3x^2+13x+5}{x(x+1)(x^2+2x+5)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+2x+5}$ とおく.
両辺を $x(x+1)(x^2+2x+5)$ 倍すると,

$$\begin{aligned}&3x^3+3x^2+13x+5\\&\quad=A(x+1)(x^2+2x+5)+Bx(x^2+2x+5)+(Cx+D)x(x+1)\end{aligned}$$

この式に $x=0$ を代入すると, $A=1$, $x=-1$ を代入すると, $B=2$. 両辺の x^3 の係数を比較すると, $A+B+C=3$ より, $C=0$. $x=1$ を代入すると, $8A+4B+C+D=12$ より, $D=-4$.

よって,

$$\frac{3x^3+3x^2+13x+5}{x(x+1)(x^2+2x+5)}=\frac{1}{x}+\frac{2}{x+1}-\frac{4}{x^2+2x+5}$$

(17) $\dfrac{x^2}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$ とおく.
両辺を $(x^2+1)^2$ 倍すると,

$$\begin{aligned}x^2&=(Ax+B)(x^2+1)+Cx+D\\&=Ax^3+Bx^2+(A+C)x+B+D\end{aligned}$$

これより, $A=0$, $B=1$, $C=0$, $D=-1$. よって,

$$\frac{x^2}{(x^2+1)^2}=\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}$$

(18) $\dfrac{2x^2-5x-4}{(x+1)^2(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{Cx+D}{x^2-x+1}$ とおく.
両辺を $(x+1)^2(x^2-x+1)$ 倍すると,

$$2x^2-5x-4=A(x+1)(x^2-x+1)+B(x^2-x+1)+(Cx+D)(x+1)^2$$

この式に $x=-1$ を代入すると, $B=1$, $x=0$ を代入すると, $A+B+D=-4$ より, $A+D=-5$. $x=1$ を代入すると, $A+2C+2D=-4$. 両辺の x^3 の係数を比較すると, $0=A+C$. この連立方程式を解くと, $A=-2$, $C=2$, $D=-3$. よって,

$$\frac{2x^2-5x-4}{(x+1)^2(x^2-x+1)}=-\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{2x-3}{x^2-x+1}$$

(19) $\dfrac{1}{(x^2+1)(x^2+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$ とおく.
両辺を $(x^2+1)(x^2+3)$ 倍すると,

$$\begin{aligned}1&=(Ax+B)(x^2+3)+(Cx+D)(x^2+1)\\&=(A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D)\end{aligned}$$

これより, $A+C=0$, $3A+C=0$, $B+D=0$, $3B+D=1$ から $A=C=0$, $B=\dfrac{1}{2}$, $D=-\dfrac{1}{2}$. よって,

$$\frac{1}{(x^2+1)(x^2+3)}=\frac{1}{2(x^2+1)}-\frac{1}{2(x^2+3)}$$

(20) $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$ とおく.
両辺を x^3+1 倍すると,

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

この式に $x=-1$ を代入すると, $A = \dfrac{1}{3}$, $x=0$ を代入すると, $A+C=1$ より, $C=\dfrac{2}{3}$. 両辺の x^2 の係数を比較すると, $0 = A+B$ より $B = -\dfrac{1}{3}$.
よって,

$$\frac{1}{x^3+1} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$$

(21) $\dfrac{x^2}{x^3+1} = \dfrac{x^2}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$ とおく.
両辺を x^3+1 倍すると,

$$x^2 = A(x^2-x+1) + (Bx+C)(x+1)$$

この式に $x=-1$ を代入すると, $A = \dfrac{1}{3}$, $x=0$ を代入すると, $A+C=0$ より, $C=-\dfrac{1}{3}$. 両辺の x^2 の係数を比較すると, $1 = A+B$ より $B = \dfrac{2}{3}$.
よって,

$$\frac{x^2}{x^3+1} = \frac{1}{3(x+1)} + \frac{2x-1}{3(x^2-x+1)}$$

(22) $\dfrac{2x+1}{x(x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$ とおく.
両辺を $x(x-1)(x+2)$ 倍すると,

$$2x+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

この式に $x=0$ を代入すると, $A=-\dfrac{1}{2}$, $x=1$ を代入すると, $B=1$, $x=-2$ を代入すると, $C=-\dfrac{1}{2}$. よって,

$$\frac{2x+1}{x(x-1)(x+2)} = -\frac{1}{2x} + \frac{1}{x-1} - \frac{1}{2(x+2)}$$

(23) $\dfrac{x^3+x^2+2x+1}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$ とおく.
両辺を $(x^2+1)^2$ 倍すると,

$$\begin{aligned} x^3+x^2+2x+1 &= (Ax+B)(x^2+1) + Cx + D \\ &= Ax^3 + Bx^2 + (A+C)x + B + D \end{aligned}$$

これより, $A=B=C=1$, $D=0$. よって,

$$\frac{x^3+x^2+2x+1}{(x^2+1)^2} = \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

(24) $\dfrac{x+1}{x^2(x^2+4x+5)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+4x+5}$ とおく.
両辺を $x^2(x^2+4x+5)$ 倍すると,

$$\begin{aligned} x+1 &= Ax(x^2+4x+5) + B(x^2+4x+5) + (Cx+D)x^2 \\ &= (A+C)x^3 + (4A+B+D)x^2 + (5A+4B)x + 5B \end{aligned}$$

これより, $B = \dfrac{1}{5}$, $A = \dfrac{1}{25}$, $D = -\dfrac{9}{25}$, $C = -\dfrac{1}{25}$. よって,

$$\frac{x+1}{x^2(x^2+4x+5)} = \frac{1}{25x} + \frac{1}{5x^2} - \frac{x+9}{25(x^2+4x+5)}$$

(25) $\dfrac{x^2+6x+11}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$ とおく.
両辺を $(x+2)^3$ 倍すると,

$$\begin{aligned} x^2+6x+11 &= A(x+2)^2 + B(x+2) + C \\ &= Ax^2 + (4A+B)x + (4A+2B+C) \end{aligned}$$

これより, $A = 1$, $B = 2$, $C = 3$. よって,

$$\frac{x^2+6x+11}{(x+2)^3} = \frac{1}{x+2} + \frac{2}{(x+2)^2} + \frac{3}{(x+2)^3}$$

(26) $\dfrac{5x^2-8}{x^4-5x^2+4} = \dfrac{5x^2-8}{(x-1)(x+1)(x-2)(x+2)}$
$= \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x-2} + \dfrac{D}{x+2}$ とおく.
両辺を x^4-5x^2+4 倍すると,

$$\begin{aligned} 5x^2-8 &= A(x+1)(x-2)(x+2) + B(x-1)(x-2)(x+2) \\ &\quad + C(x-1)(x+1)(x+2) + D(x-1)(x+1)(x-2) \end{aligned}$$

この式に $x = 1$ を代入すると, $A = \dfrac{1}{2}$, $x = -1$ を代入すると, $B = -\dfrac{1}{2}$, $x = 2$ を代入すると, $C = 1$, $x = -2$ を代入すると, $D = -1$. よって,

$$\frac{5x^2-8}{x^4-5x^2+4} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x-2} - \frac{1}{x+2}$$

(27) $\dfrac{x^3+x-1}{x^4+x^2} = \dfrac{x^3+x-1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$ とおく.
両辺を x^4+x^2 倍すると,

$$\begin{aligned} x^3+x-1 &= Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 \\ &= (A+C)x^3 + (B+D)x^2 + Ax + B \end{aligned}$$

これより, $A = 1$, $B = -1$, $C = 0$, $D = 1$. よって,

$$\frac{x^3+x-1}{x^4+x^2} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+1}$$

(28) $$\frac{x^2+4x-3}{x^3-2x^2-x+2} = \frac{x^2+4x-3}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$ とおく.
両辺を x^3-2x^2-x+2 倍すると,

$$x^2+4x-3 = A(x-1)(x-2) + B(x+1)(x-2) + C(x+1)(x-1)$$

この式に $x=-1$ を代入すると, $A=-1$, $x=1$ を代入すると, $B=-1$, $x=2$ を代入すると, $C=3$. よって,

$$\frac{x^2+4x-3}{x^3-2x^2-x+2} = -\frac{1}{x+1} - \frac{1}{x-1} + \frac{3}{x-2}$$

(29) $$\frac{x^2+5x+1}{x^3+4x^2+9x+10} = \frac{x^2+5x+1}{(x+2)(x^2+2x+5)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+5}$$
とおく. 両辺を $x^3+4x^2+9x+10$ 倍すると,

$$x^2+5x+1 = A(x^2+2x+5) + (Bx+C)(x+2)$$

この式に $x=-2$ を代入すると, $A=-1$, $x=0$ を代入すると, $5A+2C=1$ より, $C=3$. 両辺の x^2 の係数を比較すると, $1=A+B$ より $B=2$. よって,

$$\frac{x^2+5x+1}{x^3+4x^2+9x+10} = -\frac{1}{x+2} + \frac{2x+3}{x^2+2x+5}$$

(30) $$\frac{x^2-x+2}{x^3+3x^2+3x+1} = \frac{x^2-x+2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$ とおく.
両辺を x^3+3x^2+3x+1 倍すると,

$$\begin{aligned} x^2-x+2 &= A(x+1)^2 + B(x+1) + C \\ &= Ax^2 + (2A+B)x + (A+B+C) \end{aligned}$$

これより, $A=1$, $B=-3$, $C=4$. よって,

$$\frac{x^2-x+2}{x^3+3x^2+3x+1} = \frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{4}{(x+1)^3}$$

2.6.

(1) $$\frac{1}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$$ とおく.
両辺を $(x+1)(x-5)$ 倍すると,

$$1 = A(x-5) + B(x+1)$$

この式に $x=-1$ を代入すると, $A=-\frac{1}{6}$, $x=5$ を代入すると, $B=\frac{1}{6}$.
よって,

$$\frac{1}{(x+1)(x-5)} = \frac{1}{6(x-5)} - \frac{1}{6(x+1)}$$

より,

$$\int \frac{1}{(x+1)(x-5)}dx = \int \left\{ \frac{1}{6(x-5)} - \frac{1}{6(x+1)} \right\} dx$$
$$= \frac{1}{6}\log|x-5| - \frac{1}{6}\log|x+1| + C$$

(2) $\dfrac{1}{x^2-6x-7} = \dfrac{1}{(x+1)(x-7)} = \dfrac{A}{x+1} + \dfrac{B}{x-7}$ とおく.
両辺を x^2-6x-7 倍すると,

$$1 = A(x-7) + B(x+1)$$

この式に $x = -1$ を代入すると, $A = -\dfrac{1}{8}$, $x = 7$ を代入すると, $B = \dfrac{1}{8}$.
よって,

$$\frac{1}{x^2-6x-7} = \frac{1}{8(x-7)} - \frac{1}{8(x+1)}$$

より,

$$\int \frac{1}{x^2-6x-7}dx = \int \left\{ \frac{1}{8(x-7)} - \frac{1}{8(x+1)} \right\} dx$$
$$= \frac{1}{8}\log|x-7| - \frac{1}{8}\log|x+1| + C$$

(3) $\dfrac{1}{x^2+5x+6} = \dfrac{1}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$ とおく.
両辺を x^2+5x+6 倍すると,

$$1 = A(x+3) + B(x+2)$$

この式に $x = -2$ を代入すると, $A = 1$, $x = -3$ を代入すると, $B = -1$.
よって,

$$\frac{1}{x^2+5x+6} = \frac{1}{x+2} - \frac{1}{x+3}$$

より,

$$\int \frac{1}{x^2+5x+6}dx = \int \left(\frac{1}{x+2} - \frac{1}{x+3} \right) dx$$
$$= \log|x+2| - \log|x+3| + C$$

(4) $\dfrac{1}{x^2-9} = \dfrac{1}{(x-3)(x+3)} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$ とおく.
両辺を x^2-9 倍すると,

$$1 = A(x+3) + B(x-3)$$

この式に $x = 3$ を代入すると, $A = \dfrac{1}{6}$, $x = -3$ を代入すると, $B = -\dfrac{1}{6}$.
よって,

$$\frac{1}{x^2-9} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$$

より,

$$\int \frac{1}{x^2-9}dx = \int \left\{ \frac{1}{6(x-3)} - \frac{1}{6(x+3)} \right\} dx$$
$$= \frac{1}{6}\log|x-3| - \frac{1}{6}\log|x+3| + C$$

(5) $\dfrac{1}{(x+1)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+3}$ とおく.
両辺を $(x+1)(x+3)$ 倍すると,

$$1 = A(x+3) + B(x+1)$$

この式に $x = -1$ を代入すると, $A = \dfrac{1}{2}$, $x = -3$ を代入すると, $B = -\dfrac{1}{2}$.
よって,

$$\frac{1}{(x+1)(x+3)} = \frac{1}{2(x+1)} - \frac{1}{2(x+3)}$$

より,

$$\int \frac{1}{(x+1)(x+3)}dx = \int \left\{ \frac{1}{2(x+1)} - \frac{1}{2(x+3)} \right\} dx$$
$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x+3| + C$$

(6) $\dfrac{1}{x^2+6x+9} = \dfrac{1}{(x+3)^2}$ より,

$$\int \frac{1}{x^2+6x+9}dx = \int \frac{1}{(x+3)^2}dx = -\frac{1}{x+3} + C$$

(7) $\dfrac{1}{x^2-10x+25} = \dfrac{1}{(x-5)^2}$ より,

$$\int \frac{1}{x^2-10x+25}dx = \int \frac{1}{(x-5)^2}dx = -\frac{1}{x-5} + C$$

(8) $\displaystyle\int \frac{1}{x^2-6x+10}dx = \int \frac{1}{(x-3)^2+1}dx$
ここで, $x-3=t$ とおくと, $dx = dt$. よって,

$$\int \frac{1}{x^2-6x+10}dx = \int \frac{1}{t^2+1}dt$$
$$= \tan^{-1} t + C = \tan^{-1}(x-3) + C$$

(9) $\displaystyle\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx$

ここで, $x+1=t$ とおくと, $dx=dt$. よって,

$$\begin{aligned}\int \frac{1}{x^2+2x+2}dx &= \int \frac{1}{t^2+1}dt \\ &= \tan^{-1} t + C = \tan^{-1}(x+1)+C\end{aligned}$$

(10) $\displaystyle\int \frac{1}{x^2+6x+10}dx = \int \frac{1}{(x+3)^2+1}dx$

ここで, $x+3=t$ とおくと, $dx=dt$. よって,

$$\begin{aligned}\int \frac{1}{x^2+6x+10}dx &= \int \frac{1}{t^2+1}dt \\ &= \tan^{-1} t + C = \tan^{-1}(x+3)+C\end{aligned}$$

(11) $\displaystyle\frac{3x+1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$ とおく.

両辺を x^2-1 倍すると,

$$3x+1 = A(x+1)+B(x-1)$$

この式に $x=1$ を代入すると, $A=2$, $x=-1$ を代入すると, $B=1$. よって,

$$\frac{3x+1}{x^2-1} = \frac{2}{x-1} + \frac{1}{x+1}$$

より,

$$\begin{aligned}\int \frac{3x+1}{x^2-1}dx &= \int \left(\frac{2}{x-1} + \frac{1}{x+1}\right)dx \\ &= 2\log|x-1| + \log|x+1| + C\end{aligned}$$

(12) $\displaystyle\frac{x+2}{x^2-4x+3} = \frac{x+2}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$ とおく.

両辺を x^2-4x+3 倍すると,

$$x+2 = A(x-1)+B(x-3)$$

この式に $x=1$ を代入すると, $B=-\dfrac{3}{2}$, $x=3$ を代入すると, $A=\dfrac{5}{2}$. よって,

$$\frac{x+2}{x^2-4x+3} = \frac{5}{2(x-3)} - \frac{3}{2(x-1)}$$

より,

$$\begin{aligned}\int \frac{x+2}{x^2-4x+3}dx &= \int \left\{\frac{5}{2(x-3)} - \frac{3}{2(x-1)}\right\}dx \\ &= \frac{5}{2}\log|x-3| - \frac{3}{2}\log|x-1| + C\end{aligned}$$

(13) $\displaystyle\int \frac{2x+1}{x^2+x+1}dx = \int \frac{(x^2+x+1)'}{x^2+x+1}dx = \log(x^2+x+1)+C$

(14) $\displaystyle\frac{x}{x^2+2x-3} = \frac{x}{(x+3)(x-1)} = \frac{A}{x-1} + \frac{B}{x+5}$ とおく.
両辺を x^2+2x-3 倍すると,

$$x = A(x+3) + B(x-1)$$

この式に $x=1$ を代入すると, $A=\dfrac{1}{4}$, $x=-3$ を代入すると, $B=\dfrac{3}{4}$.
よって,

$$\frac{x}{x^2+2x-3} = \frac{1}{4(x-1)} + \frac{3}{4(x+3)}$$

より,

$$\begin{aligned}\int \frac{x}{x^2+2x-3}dx &= \int \left\{\frac{1}{4(x-1)} + \frac{3}{4(x+3)}\right\}dx \\ &= \frac{1}{4}\log|x-1| + \frac{3}{4}\log|x+3| + C\end{aligned}$$

(15) $\displaystyle\frac{2x-11}{2x^2-x-6} = \frac{2x-11}{(x-2)(2x+3)} = \frac{A}{x-2} + \frac{B}{2x+3}$ とおく.
両辺を $2x^2-x-6$ 倍すると,

$$2x-11 = A(2x+3) + B(x-2)$$

この式に $x=2$ を代入すると, $A=-1$, $x=-\dfrac{3}{2}$ を代入すると, $B=4$.
よって,

$$\frac{2x-11}{2x^2-x-6} = \frac{4}{2x+3} - \frac{1}{x-2}$$

より,

$$\begin{aligned}\int \frac{2x-11}{2x^2-x-6}dx &= \int \left(\frac{4}{2x+3} - \frac{1}{x-2}\right)dx \\ &= 2\log|2x+3| - \log|x+2| + C\end{aligned}$$

(16) $\displaystyle\int \frac{8x}{1+4x^2}dx = \int \frac{(1+4x^2)'}{1+4x^2}dx = \log(1+4x^2)+C$

(17) $\displaystyle\int \frac{3x+10}{x^2+6x+9}dx = \int \frac{3x+10}{(x+3)^2}dx$
ここで, $x+3=t$ とおくと, $dx=dt$. よって,

$$\begin{aligned}\int \frac{3x+10}{x^2+6x+9}dx &= \int \frac{3t+1}{t^2}dt \\ &= \int \left(\frac{3}{t}+\frac{1}{t^2}\right)dt \\ &= 3\log|t| - \frac{1}{t} + C \\ &= 3\log|x+3| - \frac{1}{x+3} + C\end{aligned}$$

(18) $\displaystyle\int \frac{x-3}{x^2-6x+10}dx = \frac{1}{2}\int \frac{(x^2-6x+10)'}{x^2-6x+10}dx = \frac{1}{2}\log(x^2-6x+10)+C$

(19) $\displaystyle\frac{2x-5}{x^2-6x-7} = \frac{2x-5}{(x-7)(x+1)} = \frac{A}{x-7}+\frac{B}{x+1}$ とおく.
両辺を x^2-6x-7 倍すると,

$$2x-5 = A(x+1)+B(x-7)$$

この式に $x=7$ を代入すると, $A=\dfrac{9}{8}$, $x=-1$ を代入すると, $B=\dfrac{7}{8}$.
よって,

$$\frac{2x-5}{x^2-6x-7} = \frac{9}{8(x-7)}+\frac{7}{8(x+1)}$$

より,

$$\begin{aligned}\int \frac{2x-5}{x^2-6x-7}dx &= \int \left\{\frac{9}{8(x-7)}+\frac{7}{8(x+1)}\right\}dx \\ &= \frac{9}{8}\log|x-7| + \frac{7}{8}\log|x+1| + C\end{aligned}$$

(20) $\displaystyle\int \frac{x}{x^2-2x+2}dx = \int \frac{x}{(x-1)^2+1}dx$
ここで, $x-1=t$ とおくと, $dx=dt$. よって,

$$\begin{aligned}\int \frac{x}{x^2-2x+2}dx &= \int \frac{t+1}{t^2+1}dt \\ &= \int \left(\frac{t}{t^2+1}+\frac{1}{t^2+1}\right)dt \\ &= \int \left(\frac{1}{2}\frac{(t^2+1)'}{t^2+1}+\frac{1}{t^2+1}\right)dt \\ &= \frac{1}{2}\log(t^2+1)+\tan^{-1}t + C \\ &= \frac{1}{2}\log(x^2-2x+2)+\tan^{-1}(x-1) + C\end{aligned}$$

(21) $\displaystyle\int \frac{3x}{4+5x^2}dx = \frac{3}{10}\int \frac{(4+5x^2)'}{4+5x^2}dx = \frac{3}{10}\log(4+5x^2) + C$

(22) $\displaystyle\int \frac{x+6}{x^2+6x+10}dx = \int \frac{x+6}{(x+3)^2+1}dx$
ここで, $x+3=t$ とおくと, $dx = dt$. よって,

$$\begin{aligned}\int \frac{x+6}{x^2+6x+10}dx &= \int \frac{t+3}{t^2+1}dt \\ &= \int \left(\frac{t}{t^2+1} + \frac{3}{t^2+1}\right)dt \\ &= \int \left(\frac{1}{2}\frac{(t^2+1)'}{t^2+1} + \frac{3}{t^2+1}\right)dt \\ &= \frac{1}{2}\log(t^2+1) + 3\tan^{-1}t + C \\ &= \frac{1}{2}\log(x^2+6x+10) + 3\tan^{-1}(x+3) + C\end{aligned}$$

(23) $\displaystyle\int \frac{x}{x^2-16}dx = \frac{1}{2}\int \frac{(x^2-16)'}{x^2-16}dx = \frac{1}{2}\log|x^2-16| + C$

(24) $\displaystyle\int \frac{x-4}{x^2-10x+25}dx = \int \frac{x-4}{(x-5)^2}dx$
ここで, $x-5=t$ とおくと, $dx = dt$. よって,

$$\begin{aligned}\int \frac{x-4}{x^2-10x+25}dx &= \int \frac{t+1}{t^2}dt \\ &= \int \left(\frac{1}{t} + \frac{1}{t^2}\right)dt \\ &= \log|t| - \frac{1}{t} + C \\ &= \log|x-5| - \frac{1}{x-5} + C\end{aligned}$$

(25) $\displaystyle\int \frac{2-x}{4x+5}dx = \int \left\{-\frac{1}{4} + \frac{13}{4(4x+5)}\right\}dx = -\frac{1}{4}x + \frac{13}{16}\log|4x+5| + C$

(26) $\displaystyle\int \frac{x^2+3}{x+1}dx = \int \left\{x-1+\frac{4}{x+1}\right\}dx = \frac{1}{2}x^2 - x + 4\log|x+1| + C$

(27) $\displaystyle\frac{2x-3}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$ とおく.
両辺を $(x+1)^2(x+2)^2$ 倍すると,

$$2x-3 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$$

この式に $x=-1$ を代入すると, $B=-5$, $x=-2$ を代入すると, $D=-7$, $x=0$ を代入すると, $2A+C=12$, 両辺の x^3 の係数を比較すると $A+C=0$.

よって, $A=12$, $C=-12$ より

$$\frac{2x-3}{(x+1)^2(x+2)^2}=\frac{12}{x+1}-\frac{5}{(x+1)^2}-\frac{12}{x+2}-\frac{7}{(x+2)^2}$$

より,

$$\int\frac{2x-3}{(x+1)^2(x+2)^2}dx=\int\left\{\frac{12}{x+1}-\frac{5}{(x+1)^2}-\frac{12}{x+2}-\frac{7}{(x+2)^2}\right\}dx$$
$$=12\log|x+1|+\frac{5}{x+1}-12\log|x+2|+\frac{7}{x+2}+C$$

(28) $\displaystyle\int\frac{x^2-x+1}{x+5}dx=\int\left(x-6+\frac{31}{x+5}\right)dx=\frac{1}{2}x^2-6x+31\log|x+5|+C$

(29) $\displaystyle\frac{x^2+x+1}{x^2-5x+6}=1+\frac{6x-5}{x^2-5x+6}$. ここで,

$$\frac{6x-5}{x^2-5x+6}=\frac{6x-5}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}$$

とおく. 両辺を x^2-5x+6 倍すると,

$$6x-5=A(x-2)+B(x-3)$$

この式に $x=3$ を代入すると, $A=13$, $x=2$ を代入すると, $B=-7$. よって,

$$\frac{6x-5}{x^2-5x+6}=\frac{13}{x-3}-\frac{7}{x-2}$$

より,

$$\int\frac{x^2+x+1}{x^2-5x+6}dx=\int\left(1+\frac{13}{x-3}-\frac{7}{x-2}\right)dx$$
$$=x+13\log|x-3|-7\log|x-2|+C$$

(30) $\displaystyle\int\frac{x^2-x+1}{x+2}dx=\int\left(x-3+\frac{7}{x+2}\right)dx=\frac{1}{2}x^2-3x+7\log|x+2|+C$

(31) $\displaystyle\frac{2x+3}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ とおく.
両辺を $(x+2)(x-1)^2$ 倍すると,

$$2x+3=A(x-1)^2+B(x+2)(x-1)+C(x+2)$$

この式に $x=-2$ を代入すると, $A=-\dfrac{1}{9}$, $x=1$ を代入すると, $C=\dfrac{5}{3}$, 両辺の x^2 の係数を比較すると, $A+B=0$ より $B=\dfrac{1}{9}$. よって,

$$\frac{2x+3}{(x+2)(x-1)^2}=-\frac{1}{9(x+2)}+\frac{1}{9(x-1)}+\frac{5}{3(x-1)^2}$$

より,

$$\int \frac{2x+3}{(x+2)(x-1)^2}dx = \int \left\{-\frac{1}{9(x+2)} + \frac{1}{9(x-1)} + \frac{5}{3(x-1)^2}\right\} dx$$
$$= -\frac{1}{9}\log|x+2| + \frac{1}{9}\log|x-1| - \frac{5}{3(x-1)} + C$$

(32) $\dfrac{1}{(x+1)(x-2)(x+5)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x+5}$ とおく.
両辺を $(x+1)(x-2)(x+5)$ 倍すると,

$$1 = A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)$$

この式に $x=-1$ を代入すると, $A = -\dfrac{1}{12}$, $x=2$ を代入すると, $B = \dfrac{1}{21}$, $x=-5$ を代入すると, $C = \dfrac{1}{28}$. よって,

$$\frac{1}{(x+1)(x-2)(x+5)} = -\frac{1}{12(x+1)} + \frac{1}{21(x-2)} + \frac{1}{28(x+5)}$$

より,

$$\int \frac{1}{(x+1)(x-2)(x+5)}dx = \int \left\{-\frac{1}{12(x+1)} + \frac{1}{21(x-2)} + \frac{1}{28(x+5)}\right\} dx$$
$$= -\frac{1}{12}\log|x+1| + \frac{1}{21}\log|x-2| + \frac{1}{28}\log|x+5| + C$$

(33) $$\int \frac{x^4+x^3+x^2-4x-5}{x^2-x-1}dx = \int \left(x^2+2x+4+\frac{2x-1}{x^2-x-1}\right)dx$$
$$= \frac{1}{3}x^3 + x^2 + 4x + \log|x^2-x-1| + C$$

(34) $\dfrac{x+1}{(x-1)(x^2+2x+2)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+2x+2}$ とおく.
両辺を $(x-1)(x^2+2x+2)$ 倍すると,

$$x+1 = A(x^2+2x+2) + (Bx+C)(x-1)$$

この式に $x=1$ を代入すると, $A = \dfrac{2}{5}$, $x=0$ を代入すると, $C = -\dfrac{1}{5}$, 両辺の x^2 の係数を比較すると, $0 = A+B$ から $B = -\dfrac{2}{5}$. よって,

$$\frac{x+1}{(x-1)(x^2+2x+2)} = \frac{2}{5(x-1)} - \frac{2x+1}{5(x^2+2x+2)}$$

より,

$$\int \frac{x+1}{(x-1)(x^2+2x+2)}dx = \int \left\{\frac{2}{5(x-1)} - \frac{2x+1}{5(x^2+2x+2)}\right\} dx$$
$$= \frac{2}{5}\log|x-1| - \frac{1}{5}\int \frac{2x+1}{(x+1)^2+1}dx$$

ここで, $\displaystyle\int \frac{2x+1}{(x+1)^2+1}dx$ において, $x+1=t$ とおく. $dx=dt$ より,

$$\begin{aligned}\int \frac{2x+1}{(x+1)^2+1}dx &= \int \frac{2t-1}{t^2+1}dt \\ &= \int \frac{2t}{t^2+1}dt - \int \frac{1}{t^2+1}dt \\ &= \log(t^2+1) - \tan^{-1} t + C \\ &= \log(x^2+2x+2) - \tan^{-1}(x+1) + C\end{aligned}$$

ゆえに,

$$(\text{与式}) = \frac{2}{5}\log|x-1| - \frac{1}{5}\log(x^2+2x+2) + \frac{1}{5}\tan^{-1}(x+1) + C$$

(35) $\displaystyle\frac{x^3+x-1}{x^4-1} = \frac{x^3+x-1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$ とおく.
両辺を x^4-1 倍すると,

$$x^3+x-1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

この式に $x=1$ を代入すると, $A=\dfrac{1}{4}$, $x=-1$ を代入すると, $B=\dfrac{3}{4}$, $x=0$ を代入すると, $-1=A-B-D$ から $D=\dfrac{1}{2}$, 両辺の x^3 の係数を比較すると, $1=A+B+C$ から $C=0$. よって,

$$\frac{x^3+x-1}{x^4-1} = \frac{1}{4(x-1)} + \frac{3}{4(x+1)} + \frac{1}{2(x^2+1)}$$

より,

$$\begin{aligned}\int \frac{x^3+x-1}{x^4-1}dx &= \int \left\{\frac{1}{4(x-1)} + \frac{3}{4(x+1)} + \frac{1}{2(x^2+1)}\right\}dx \\ &= \frac{1}{4}\log|x-1| + \frac{3}{4}\log|x+1| + \frac{1}{2}\tan^{-1} x + C\end{aligned}$$

(36) $\displaystyle\frac{x+2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$ とおく.
両辺を $(x-1)(x^2+1)^2$ 倍すると,

$$\begin{aligned}x+2 &= A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \\ &= (A+B)x^4 - (B-C)x^3 + (2A+B-C+D)x^2 \\ &\qquad + (-B+C-D+E)x + (A-C-E)\end{aligned}$$

これより,

$$\begin{cases} A+B=0 \\ B-C=0 \\ 2A+B-C+D=0 \\ -B+C-D+E=1 \\ A-C-E=2 \end{cases}$$

これを解くと, $A=\dfrac{3}{4}$, $B=C=-\dfrac{3}{4}$, $D=-\dfrac{3}{2}$, $E=-\dfrac{1}{2}$. よって,

$$\int \frac{x+2}{(x-1)(x^2+1)^2}dx = \int \left\{ \frac{3}{4(x-1)} - \frac{3x+3}{4(x^2+1)} - \frac{3x+1}{2(x^2+1)^2} \right\} dx$$
$$= \frac{3}{4}\log|x-1| - \int \left\{ \frac{3x}{4(x^2+1)} + \frac{3}{4(x^2+1)} + \frac{3x+1}{2(x^2+1)^2} \right\} dx$$
$$= \frac{3}{4}\log|x-1| - \frac{3}{8}\log(x^2+1) - \frac{3}{4}\tan^{-1}x$$
$$- \frac{3}{2}\int \frac{x}{(x^2+1)^2}dx - \frac{1}{2}\int \frac{1}{(x^2+1)^2}dx$$

ここで, $\displaystyle\int \frac{x}{(x^2+1)^2}dx$ において, $x^2+1=t$ とおくと, $xdx=\dfrac{1}{2}dt$. よって,

$$\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int \frac{1}{t^2}dt = -\frac{1}{2t} + C = -\frac{1}{2(x^2+1)} + C$$

一方, 定理 2.6 より

$$\int \frac{1}{(x^2+1)^2}dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + C$$

よって,

$$(\text{与式}) = \frac{3}{4}\log|x-1| - \frac{3}{8}\log(x^2+1) - \frac{3}{4}\tan^{-1}x$$
$$- \frac{3}{2}\int \frac{x}{(x^2+1)^2}dx - \frac{1}{2}\int \frac{1}{(x^2+1)^2}dx$$
$$= \frac{3}{4}\log|x-1| - \frac{3}{8}\log(x^2+1) - \frac{3}{4}\tan^{-1}x$$
$$+ \frac{3}{4(x^2+1)} - \frac{x}{4(x^2+1)} - \frac{1}{4}\tan^{-1}x + C$$
$$= \frac{3}{4}\log|x-1| - \frac{3}{8}\log(x^2+1) - \frac{3}{4}\tan^{-1}x - \frac{x-3}{4(x^2+1)} - \frac{1}{4}\tan^{-1}x + C$$

2.7.

(1) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち $\sin x dx = -dt$. よって,

$$\int (\cos^3 x - 1)\sin x dx = \int (1-t^3)dt = t - \frac{1}{4}t^4 + C = -\frac{1}{4}\cos^4 x + \cos x + C$$

(2) $\sin x = t$ とおくと, $\cos x dx = dt$. よって,

$$\int \frac{\cos x}{\sin^2 x + 1}dx = \int \frac{1}{t^2+1}dt = \tan^{-1}t + C = \tan^{-1}(\sin x) + C$$

(3) $\sin x = t$ とおくと, $\cos x dx = dt$. よって,

$$\begin{aligned}\int (\sin^2 x + \sin x + 1)\cos x dx &= \int (t^2 + t + 1) dt \\ &= \frac{1}{3}t^3 + \frac{1}{2}t^2 + t + C \\ &= \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + \sin x + C\end{aligned}$$

(4) $\sin x = t$ とおくと, $\cos x dx = dt$. よって,

$$\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{t^2} dt = -\frac{1}{t} + C = -\frac{1}{\sin x} + C$$

(5) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち, $\sin x dx = -dt$. よって,

$$\int \{\sin(\cos x)\}\sin x dx = -\int \sin t dt = \cos t + C = \cos(\cos x) + C$$

(6) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち, $\sin x dx = -dt$. よって,

$$\begin{aligned}\int (\cos^3 x - 5\cos x + 7)\sin x dx &= -\int (t^3 - 5t + 7) dt \\ &= -\left(\frac{1}{4}t^4 - \frac{5}{2}t^2 + 7t\right) + C \\ &= -\frac{1}{4}\cos^4 x + \frac{5}{2}\cos^2 x - 7\cos x + C\end{aligned}$$

(7) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち $\sin x dx = -dt$. よって,

$$\int \frac{\sin x}{\cos x + 1} dx = -\int \frac{1}{t+1} dt = -\log|t+1| + C = -\log(1 + \cos x) + C$$

(8) $\cos x = t$ とおくと, $-\sin x dx = dt$, すなわち $\sin x dx = -dt$. よって,

$$\begin{aligned}\int \sin 2x \cos x dx &= 2\int \sin x \cos^2 x dx \\ &= -2\int t^2 dt \\ &= -\frac{2}{3}t^3 + C \\ &= -\frac{2}{3}\cos^3 x + C\end{aligned}$$

(9) $\tan x = t$ とおくと,

$$\frac{1}{\cos^2 x} dx = dt \iff (1 + \tan^2 x) dx = dt \iff dx = \frac{1}{1+t^2} dt$$

よって,

$$
\begin{aligned}
\int (\tan^3 x + 2\tan^2 x + 1)dx &= \int \frac{t^3 + 2t^2 + 1}{t^2 + 1} dt \\
&= \int \left(t + 2 - \frac{t}{t^2+1} - \frac{1}{t^2+1} \right) dt \\
&= \frac{1}{2}t^2 + 2t - \frac{1}{2}\log(t^2+1) - \tan^{-1} t + C \\
&= \frac{1}{2}\tan^2 x + 2\tan x - \frac{1}{2}\log(\tan^2 x + 1) - \tan^{-1}(\tan x) + C \\
&= \frac{1}{2}\tan^2 x + 2\tan x + \log|\cos x| - \tan^{-1}(\tan x) + C
\end{aligned}
$$

(10) $\tan x = t$ とおくと,

$$
\frac{1}{\cos^2 x} dx = dt \iff (1 + \tan^2 x)dx = dt \iff dx = \frac{1}{1+t^2} dt
$$

よって,

$$
\begin{aligned}
\int \frac{1}{\tan^2 x} dx &= \int \frac{1}{t^2(1+t^2)} dt \\
&= \int \left(\frac{1}{t^2} - \frac{1}{t^2+1} \right) dt \\
&= -\frac{1}{t} - \tan^{-1} t + C \\
&= -\frac{1}{\tan x} - \tan^{-1}(\tan x) + C
\end{aligned}
$$

(11) $\tan x = t$ とおくと,

$$
\frac{1}{\cos^2 x} dx = dt \iff (1 + \tan^2 x)dx = dt \iff dx = \frac{1}{1+t^2} dt
$$

よって,

$$
\begin{aligned}
\int \frac{1}{2\tan^2 x - 1} dx &= \int \frac{1}{(2t^2-1)(t^2+1)} dx \\
&= \int \left\{ \frac{1}{3(\sqrt{2}t - 1)} - \frac{1}{3(\sqrt{2}t+1)} - \frac{1}{3(t^2+1)} \right\} dt \\
&= \frac{1}{3\sqrt{2}} \log|\sqrt{2}t - 1| - \frac{1}{3\sqrt{2}} \log|\sqrt{2}t + 1| - \frac{1}{3}\tan^{-1} t + C \\
&= \frac{1}{3\sqrt{2}} \log\left| \frac{\sqrt{2}\tan x - 1}{\sqrt{2}\tan x + 1} \right| - \frac{1}{3}\tan^{-1}(\tan x) + C
\end{aligned}
$$

(12) $\displaystyle\int(\sin x+\cos x)^2dx=\int(1+2\sin x\cos x)dx$

$$=\int(1+\sin 2x)dx=x-\frac{1}{2}\cos 2x+C.$$

(13) $\displaystyle\int\frac{1}{\sin x}dx=\int\frac{\sin x}{\sin^2 x}dx=\int\frac{\sin x}{1-\cos^2 x}dx.$
$\cos x=t$ とおくと, $-\sin x dx=dt$. よって,

$$\begin{aligned}\int\frac{1}{\sin x}dx&=\int\frac{1}{t^2-1}dt\\&=\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\\&=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2}\log\left|\frac{\cos x-1}{\cos x+1}\right|+C\end{aligned}$$

(14) $\tan\dfrac{x}{2}=t$ とおく. $\cos x=\dfrac{1-t^2}{1+t^2}$, $dx=\dfrac{2}{1+t^2}dt$. よって,

$$\begin{aligned}\int\frac{1}{1+\cos x}dx&=\int\frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt\\&=\int dt\\&=t+C=\tan\frac{x}{2}+C\end{aligned}$$

(15) $\tan\dfrac{x}{2}=t$ とおく. $\sin x=\dfrac{2t}{1+t^2}$, $\cos x=\dfrac{1-t^2}{1+t^2}$, $dx=\dfrac{2}{1+t^2}dt$. よって,

$$\begin{aligned}\int\frac{2+\sin x}{(1+\cos x)\sin x}dx&=\int\frac{2+\frac{2t}{1+t^2}}{(1+\frac{1-t^2}{1+t^2})\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}dt\\&=\int\left(t+1+\frac{1}{t}\right)dt\\&=\frac{1}{2}t^2+t+\log|t|+C\\&=\frac{1}{2}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\log\left|\tan\frac{x}{2}\right|+C\end{aligned}$$

(16) $\displaystyle\int\frac{\cos x-\sin x}{\cos x+\sin x}dx=\int\frac{(\cos x+\sin x)'}{\cos x+\sin x}dx=\log|\cos x+\sin x|+C$

(17) $\tan\dfrac{x}{2}=t$ とおく. $\sin x=\dfrac{2t}{1+t^2}$, $dx=\dfrac{2}{1+t^2}dt$. よって,

$$\begin{aligned}
\int\frac{1}{2+\sin x}dx &= \int\frac{1}{2+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}dt \\
&= \int\frac{1}{t^2+t+1}dt \\
&= \int\frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}dt \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}+C \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}+C
\end{aligned}$$

(18) $\displaystyle\int\frac{\cos x-1}{\sin x+1}dx=\int\left(\frac{\cos x}{\sin x+1}-\frac{1}{\sin x+1}\right)dx=\log(\sin x+1)-\int\frac{1}{\sin x+1}dx$

ここで, $\displaystyle\int\frac{1}{\sin x+1}dx$ において $\tan\dfrac{x}{2}=t$ とおく. $\sin x=\dfrac{2t}{1+t^2}$, $dx=\dfrac{2}{1+t^2}dt$. よって,

$$\begin{aligned}
\int\frac{1}{\sin x+1}dx &= \int\frac{1}{\frac{2t}{1+t^2}+1}\cdot\frac{2}{1+t^2}dt \\
&= \int\frac{2}{(t+1)^2}dt \\
&= -\frac{2}{t+1}+C=-\frac{2}{\tan\frac{x}{2}+1}+C
\end{aligned}$$

よって,

$$\int\frac{\cos x-1}{\sin x+1}dx=\log(\sin x+1)+\frac{2}{\tan\frac{x}{2}+1}+C$$

2.8.

(1) $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx.$
$x^{\frac{1}{6}} = t$ とおくと, $x = t^6$, $dx = 6t^5dt$. よって,

$$\begin{aligned}\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx &= \int \frac{6t^5}{t^3+t^2}dt \\ &= \int \frac{6t^3}{t+1}dt \\ &= 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)dt \\ &= 6\left(\frac{1}{3}t^3 - \frac{1}{2}t^2 + t - \log|t+1|\right) + C \\ &= 2t^3 - 3t^2 + 6t - 6\log|t+1| + C\end{aligned}$$

(2) $\sqrt{2x+3} = t$ とおくと, $x = \dfrac{t^2-3}{2}$, $dx = tdt$. よって,

$$\begin{aligned}\int \frac{1}{\sqrt{2x+3}}dx &= \int \frac{t}{t}dt \\ &= \int dt \\ &= t + C = \sqrt{2x+3} + C\end{aligned}$$

(3) $\sqrt[3]{x-1} = t$ とおく. $x = t^3+1$, $dx = 3t^2dt$. よって,

$$\begin{aligned}\int \frac{x}{\sqrt[3]{x-1}}dx &= \int \frac{t^3+1}{t}\cdot 3t^2dt \\ &= 3\int (t^4+t)dt \\ &= \frac{3}{5}t^5 + \frac{3}{2}t^2 + C = \frac{3}{5}(x-1)^{\frac{5}{3}} + \frac{3}{2}(x-1)^{\frac{2}{3}} + C\end{aligned}$$

(4) $\sqrt{x^2-1} = t - x$ とおく. $x = \dfrac{t^2+1}{2t} = \dfrac{t}{2} + \dfrac{1}{2t}$, $dx = \left(\dfrac{1}{2} - \dfrac{1}{2t^2}\right)dt = \dfrac{t^2-1}{2t^2}dt.$ また,

$$\sqrt{x^2-1} = t - x = t - \frac{t^2+1}{2t} = \frac{t^2-1}{2t}$$

よって,

$$\int \frac{1}{x^2\sqrt{x^2-1}}dx = \int \frac{1}{(\frac{t^2+1}{2t})^2\frac{t^2-1}{2t}} \cdot \frac{t^2-1}{2t^2}dt$$
$$= \int \frac{4t}{(t^2+1)^2}dt$$

ここで, $t^2+1=u$ とおくと, $2tdt=du$. よって,

$$(\text{与式}) = \int \frac{2}{u^2}du$$
$$= -\frac{2}{u}+C$$
$$= -\frac{2}{t^2+1}+C$$
$$= -\frac{2}{(x+\sqrt{x^2-1})^2+1}+C$$
$$= -\frac{1}{x^2+x\sqrt{x^2-1}}+C$$

(5) $\sqrt{x^2-4} = t-x$ とおく. $x = \frac{t^2+4}{2t} = \frac{t}{2}+\frac{2}{t}$, $dx = \left(\frac{1}{2}-\frac{2}{t^2}\right)dt = \frac{t^2-4}{2t^2}dt.$ また,

$$\sqrt{x^2-4} = t-x = t-\frac{t^2+4}{2t} = \frac{t^2-4}{2t}$$

よって,

$$\int \sqrt{x^2-4}dx = \int \frac{t^2-4}{2t}\cdot\frac{t^2-4}{2t^2}dt$$
$$= \frac{1}{4}\int \frac{t^4-8t^2+16}{t^3}dt$$
$$= \frac{1}{4}\int \left(t-\frac{8}{t}+\frac{16}{t^3}\right)dt$$
$$= \frac{1}{8}t^2-2\log|t|-\frac{2}{t^2}+C$$
$$= \frac{1}{8}(\sqrt{x^2+4}+x)^2-2\log|\sqrt{x^2+4}+x|-\frac{2}{(\sqrt{x^2+4}+x)^2}+C$$
$$= \frac{1}{8}(2x^2+2x\sqrt{x^2-4}-4)-2\log|\sqrt{x^2+4}+x|-\frac{1}{8}(2x^2-2x\sqrt{x^2-4}-4)+C$$
$$= \frac{1}{2}x\sqrt{x^2-4}-2\log|\sqrt{x^2+4}+x|+C$$

(6) $(2x+1)^{\frac{3}{4}} = t$ とおく. $x = \dfrac{t^{\frac{4}{3}} - 1}{2}$, $dx = \dfrac{2}{3}t^{\frac{1}{3}}dt$. よって,

$$\begin{aligned}
\int \frac{x^2}{(2x+1)^{\frac{3}{4}}}dx &= \int \frac{(\frac{t^{\frac{4}{3}}-1}{2})^2}{t} \cdot \frac{2}{3}t^{\frac{1}{3}}dt \\
&= \int \frac{t^{\frac{8}{3}} - 2t^{\frac{4}{3}} + 1}{12t} \cdot 2t^{\frac{1}{3}}dt \\
&= \int \frac{t^3 - t^{\frac{5}{3}} + t^{\frac{1}{3}}}{6t}dt \\
&= \frac{1}{6}\int \left(t^2 - t^{\frac{2}{3}} + t^{\frac{-2}{3}}\right)dt \\
&= \frac{1}{6}\left(\frac{1}{3}t^3 - \frac{3}{5}t^{\frac{5}{3}} + 3t^{\frac{1}{3}}\right) + C \\
&= \frac{1}{18}(2x+1)^{\frac{9}{4}} - \frac{1}{10}(2x+1)^{\frac{5}{4}} + \frac{1}{2}(2x+1)^{\frac{1}{4}} + C \\
&= \frac{1}{90}(2x+1)^{\frac{1}{4}}\left\{5(2x+1)^2 - 9(2x+1) + 45\right\} + C \\
&= \frac{1}{90}(2x+1)^{\frac{1}{4}}(20x^2 + 2x + 41) + C
\end{aligned}$$

(7) $\sqrt{x+1} = t$ とおく. $x = t^2 - 1$, $dx = 2tdt$. よって,

$$\begin{aligned}
\int x\sqrt{x+1}dx &= \int (t^2-1)t \cdot 2tdt \\
&= 2\int (t^4 - t^2)dt \\
&= \frac{2}{5}t^5 - \frac{2}{3}t^3 + C \\
&= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \\
&= \frac{2}{15}(x+1)^{\frac{3}{2}}\{3(x+1) - 5\} \\
&= \frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2) + C
\end{aligned}$$

(8) $\displaystyle\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{1}{\sqrt{x(1-x)}}dx = \int \frac{1}{x\sqrt{\frac{1-x}{x}}}dx$

$\sqrt{\dfrac{1-x}{x}} = t$ とおく．$x = \dfrac{1}{t^2+1}$, $dx = \dfrac{-2t}{(t^2+1)^2}dt$. よって，

$$\begin{aligned}\int \frac{1}{\sqrt{x-x^2}}dx &= \int \frac{1}{\frac{1}{t^2+1}\cdot t}\cdot\frac{-2t}{t^2+1}dt \\ &= -2\int dt \\ &= -2t + C = -2\sqrt{\frac{1-x}{x}} + C\end{aligned}$$

(9) $\sqrt{x^2+9} = t - x$ とおく．$x = \dfrac{t^2-9}{2t} = \dfrac{t}{2} - \dfrac{9}{2t}$, $dx = \left(\dfrac{1}{2} + \dfrac{9}{2t^2}\right)dt = \dfrac{t^2+9}{2t^2}dt$. また，

$$\sqrt{x^2+9} = t - x = t - \frac{t^2-9}{2t} = \frac{t^2+9}{2t}$$

よって，

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2+9}}dx &= \int \frac{1}{\frac{t^2-9}{2t}\frac{t^2+9}{2t}}\cdot\frac{t^2+9}{2t^2}dt \\ &= 2\int \frac{1}{t^2-9}dt \\ &= \frac{1}{3}\int\left(\frac{1}{t-3} - \frac{1}{t+3}\right)dt \\ &= \frac{1}{3}\log|t-3| - \frac{1}{3}\log|t+3| + C \\ &= \frac{1}{3}\log|\sqrt{x^2+9}+x-3| - \frac{1}{3}\log|\sqrt{x^2+9}+x+3| + C\end{aligned}$$

(10) $\sqrt[5]{2x+1} = t$ とおく．$x = \dfrac{t^5-1}{2}$, $dx = \dfrac{5}{2}t^4dt$. よって，

$$\begin{aligned}\int \frac{1}{\sqrt[5]{2x+1}}dx &= \int \frac{1}{t}\cdot\frac{5}{2}t^4dt \\ &= \frac{5}{2}\int t^3dt \\ &= \frac{5}{8}t^4 + C = \frac{5}{8}(2x+1)^{\frac{4}{5}} + C\end{aligned}$$

(11) $\displaystyle\int \frac{1}{\sqrt{x+2}+\sqrt{x}}dx = \frac{1}{2}\int(\sqrt{x+2}-\sqrt{x})dx$

$$= \frac{1}{2}\left\{\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right\} + C$$
$$= \frac{1}{3}\left\{(x+2)^{\frac{3}{2}} - x^{\frac{3}{2}}\right\} + C$$

(12) $\sqrt{x^2+1} = t - x$ とおく．$x = \dfrac{t^2-1}{2t} = \dfrac{t}{2} - \dfrac{1}{2t}$, $dx = \left(\dfrac{1}{2} + \dfrac{1}{2t^2}\right)dt = \dfrac{t^2+1}{2t^2}dt$. また,

$$\sqrt{x^2+1} = t - x = t - \frac{t^2-1}{2t} = \frac{t^2+1}{2t}$$

よって,

$$\int \frac{1}{x\sqrt{x^2+1}}dx = \int \frac{1}{\frac{t^2-1}{2t}\frac{t^2+1}{2t}} \cdot \frac{t^2+1}{2t^2}dt$$
$$= \int \frac{2}{t^2-1}dt$$
$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$$
$$= \log|t-1| - \log|t+1| + C$$
$$= \log|\sqrt{x^2+1}+x-1| - \log|\sqrt{x^2+1}+x+1| + C$$

(13) $\sqrt{1-x} = t$ とおく．$x = 1-t^2$, $dx = -2tdt$. よって,

$$\int \frac{1}{x\sqrt{1-x}}dx = \int \frac{1}{(1-t^2)t}\cdot(-2t)dt$$
$$= \int \frac{2}{t^2-1}dt$$
$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$$
$$= \log|t-1| - \log|t+1| + C$$
$$= \log|\sqrt{1-x}-1| - \log|\sqrt{1-x}+1| + C$$

(14) $\sqrt{x^2+x+1} = t - x$ とおく．$x = \dfrac{t^2-1}{2t+1} = \dfrac{1}{2}t - \dfrac{1}{4} - \dfrac{3}{4(2t+1)}$, $dx = \left\{\dfrac{1}{2} + \dfrac{3}{2(2t+1)^2}\right\}dt = \dfrac{2(t^2+t+1)}{(2t+1)^2}dt$. また,

$$\sqrt{x^2+x+1} = t - x = t - \frac{t^2-1}{2t+1} = \frac{t^2+t+1}{2t+1}$$

よって,

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+x+1}}dx &= \int \frac{1}{\frac{t^2-1}{2t+1}\frac{t^2+t+1}{2t+1}} \cdot \frac{2(t^2+t+1)}{(2t+1)^2}dt \\
&= \int \frac{2}{t^2-1}dt \\
&= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt \\
&= \log|t-1| - \log|t+1| + C \\
&= \log|\sqrt{x^2+x+1}+x-1| - \log|\sqrt{x^2+x+1}+x+1| + C
\end{aligned}$$

(15) $\displaystyle\int \frac{1}{1+\sqrt{1-x^2}}dx = \int \frac{1}{1+(1-x)\sqrt{\frac{1+x}{1-x}}}dx$

ここで, $\sqrt{\dfrac{1+x}{1-x}} = t$ とおく. $x = \dfrac{t^2-1}{t^2+1} = 1 - \dfrac{2}{t^2+1}$, $dx = \dfrac{4t}{(t^2+1)^2}dt$.
また,

$$1 - x = 1 - \left(1 - \frac{2}{t^2+1}\right) = \frac{2}{t^2+1}$$

よって,

$$\begin{aligned}
\int \frac{1}{1+\sqrt{1-x^2}}dx &= \int \frac{1}{1+(1-x)\sqrt{\frac{1+x}{1-x}}}dx \\
&= \int \frac{1}{1+\frac{2}{t^2+1}t} \cdot \frac{4t}{(t^2+1)^2}dt \\
&= \int \frac{4t}{(t^2+1)(t+1)^2}dt
\end{aligned}$$

ここで,

$$\frac{4t}{(t^2+1)(t+1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{Ct+D}{t^2+1}$$

とおく. 両辺を $(t^2+1)(t+1)^2$ 倍すると,

$$4t = A(t+1)(t^2+1) + B(t^2+1) + (Ct+D)(t+1)^2$$

両辺に $t=-1$ を代入すると, $B=-2$, $t=0$ を代入すると $A+B+D=0$ より $A+D=2$. 両辺の t^3 の係数を比較すると $A+C=0$. $t=1$ を代入すると $4A+2B+4C+4D=4$ より $A+C+D=2$. これより, $A=0$, $C=0$,

$D = 2$. よって,

$$\begin{aligned}
(\text{与式}) &= 2\int\left(\frac{1}{t^2+1} - \frac{1}{(t+1)^2}\right)dt \\
&= 2\tan^{-1} t + \frac{2}{t+1} + C \\
&= 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} + \frac{2}{\sqrt{\frac{1+x}{1-x}}+1} + C \\
&= 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} + \frac{2\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} + C \\
&= 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} + \frac{\sqrt{1-x^2}-1+x}{x} + C
\end{aligned}$$

(16) $\sqrt[n]{1+x} = t$ とおく. $x = t^n - 1$, $dx = nt^{n-1}dt$. よって,

$$\begin{aligned}
\int x\sqrt[n]{1+x}dx &= \int (t^n-1)t \cdot nt^{n-1}dt \\
&= n\int (t^{2n} - t^n)dt \\
&= \frac{n}{2n+1}t^{2n+1} - \frac{n}{n+1}t^{n+1} + C \\
&= \frac{n}{2n+1}(1+x)^{\frac{2n+1}{n}} - \frac{n}{n+1}(1+x)^{\frac{n+1}{n}} + C \\
&= \frac{n}{(n+1)(2n+1)}(1+x)^{\frac{n+1}{n}}\left\{(n+1)(1+x) - (2n+1)\right\} + C \\
&= \frac{n}{(n+1)(2n+1)}(1+x)^{\frac{n+1}{n}}\left\{(n+1)x - n\right\} + C
\end{aligned}$$

(17) $\sqrt{x+2} = t$ とおく. $x = t^2 - 2$, $dx = 2tdt$. よって,

$$\begin{aligned}
\int \frac{1}{1+\sqrt{x+2}}dx &= \int \frac{1}{1+t}\cdot 2tdt \\
&= 2\int\left(1 - \frac{1}{t+1}\right)dt \\
&= 2t - 2\log|t+1| + C \\
&= 2\sqrt{x+2} - 2\log(\sqrt{x+2}+1) + C
\end{aligned}$$

(18) $\sqrt{x+7}=t$ とおく． $x=t^2-7$, $dx=2tdt$. よって，

$$\begin{aligned}
\int \frac{x}{6+x\sqrt{x+7}}dx &= \int \frac{t^2-7}{6+(t^2-7)t}\cdot 2tdt \\
&= \int \frac{2t^3-14t}{t^3-7t+6}dt \\
&= \int \left(2-\frac{12}{t^3-7t+6}\right)dt \\
&= \int \left\{2-\frac{12}{(t-1)(t-2)(t+3)}\right\}dt \\
&= \int \left\{2-\frac{12}{(t-1)(t-2)(t+3)}\right\}dt \\
&= \int \left\{2+\frac{3}{t-1}-\frac{12}{5(t-2)}-\frac{3}{5(t+3)}\right\}dt \\
&= 2t+3\log|t-1|-\frac{12}{5}\log|t-2|-\frac{3}{5}\log|t+3|+C \\
&= 2\sqrt{x+7}+3\log|\sqrt{x+7}-1| \\
&\qquad -\frac{12}{5}\log|\sqrt{x+7}-2|-\frac{3}{5}\log|\sqrt{x+7}+3|+C
\end{aligned}$$

(19) $\sqrt{x^2-1}=t-x$ とおく． $x=\dfrac{t^2+1}{2t}=\dfrac{t}{2}+\dfrac{1}{2t}$, $dx=\left(\dfrac{1}{2}-\dfrac{1}{2t^2}\right)dt=\dfrac{t^2-1}{2t^2}dt$. また，

$$\sqrt{x^2-1}=t-x=t-\left(\frac{t}{2}+\frac{1}{2t}\right)=\frac{t^2-1}{2t}$$

よって，

$$\begin{aligned}
\int \sqrt{x^2-1}dx &= \int \frac{t^2-1}{2t}\cdot\frac{t^2-1}{2t^2}dt \\
&= \int \frac{t^4-2t^2+1}{4t^3}dt \\
&= \int \left(\frac{t}{4}-\frac{1}{2t}+\frac{1}{4t^3}\right)dt \\
&= \frac{1}{8}t^2-\frac{1}{2}\log|t|-\frac{1}{8t^2}+C \\
&= \frac{1}{8}(x+\sqrt{x^2-1})^2-\frac{1}{2}\log|x+\sqrt{x^2-1}|-\frac{1}{8(x+\sqrt{x^2-1})^2}+C \\
&= \frac{1}{8}(x+\sqrt{x^2-1})^2-\frac{1}{2}\log|x+\sqrt{x^2-1}|-\frac{(x-\sqrt{x^2-1})^2}{8}+C \\
&= \frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\log|x+\sqrt{x^2-1}|+C
\end{aligned}$$

(20) $4-x^2=t$ とおくと, $-2xdx=dt$. よって,

$$\int x\sqrt{4-x^2}dx=-\frac{1}{2}\int\sqrt{t}dt=-\frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}}+C=\frac{-1}{3}(4-x^2)^{\frac{3}{2}}+C$$

2.9.

(1) $$\int_1^4\left(x^2+\frac{1}{x^2}+\frac{1}{x^{\frac{3}{2}}}\right)dx=\int_1^4\left(x^2+x^{-2}+x^{-\frac{3}{2}}\right)dx$$
$$=\left[\frac{1}{3}x^3-x^{-1}-2x^{-\frac{1}{2}}\right]_1^4$$
$$=\left(\frac{64}{3}-\frac{1}{4}-\frac{2}{2}\right)-\left(\frac{1}{3}-1-2\right)=\frac{209}{12}$$

(2) $$\int_2^{e^2+1}\frac{1}{x-1}dx=[\log|x-1|]_2^{e^2+1}=\log e^2-\log 1=2$$

(3) $$\int_1^4\frac{2x^{\frac{3}{2}}+3x^2-\sqrt{x}+1}{x}dx=\int_1^4\left(2x^{\frac{1}{2}}+3x-x^{-\frac{1}{2}}+x^{-1}\right)dx$$
$$=\left[\frac{4}{3}x^{\frac{3}{2}}+\frac{3}{2}x^2-2x^{\frac{1}{2}}+\log|x|\right]_1^4$$
$$=\left(\frac{32}{3}+\frac{48}{2}-4+\log 4\right)-\left(\frac{4}{3}+\frac{3}{2}-2\right)$$
$$=\frac{179}{6}+2\log 2$$

(4) $$\int_0^1\left(\sqrt{x}-1\right)^2dx=\int_0^1(x-2\sqrt{x}+1)dx$$
$$=\int_0^1(x-2x^{\frac{1}{2}}+1)dx$$
$$=\left[\frac{1}{2}x^2-\frac{4}{3}x^{\frac{3}{2}}+x\right]_0^1=\left(\frac{1}{2}-\frac{4}{3}+1\right)=\frac{1}{6}$$

(5) $$\int_0^1\frac{1}{\sqrt[3]{x+1}}dx=\int_0^1(x+1)^{-\frac{1}{3}}dx=\left[\frac{3}{2}(x+1)^{\frac{2}{3}}\right]_0^1=\frac{3}{2}2^{\frac{2}{3}}-\frac{3}{2}=\frac{3}{\sqrt[3]{2}}-\frac{3}{2}$$

(6) $$\int_0^1\frac{3x^3+2x^2-x+4}{x+1}dx=\int_0^1\left(3x^2-x+\frac{4}{x+1}\right)dx$$
$$=\left[x^3-\frac{1}{2}x^2+4\log|x+1|\right]_0^1=\frac{1}{2}+4\log 2$$

(7) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(\sin t-\cos t)dt=[-\cos t-\sin t]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=0$$

(8) $\displaystyle\int_0^1 \frac{1}{1+t^2}dt = \left[\tan^{-1} t\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$

(9) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx = \left[\sin^{-1} x\right]_0^{\frac{1}{2}} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$

(10) $\displaystyle\int_0^{\pi} \cos^2 \frac{\theta}{2}d\theta = \int_0^{\pi} \frac{1+\cos\theta}{2}d\theta = \left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\right]_0^{\pi} = \frac{\pi}{2}$

(11)
$$\begin{aligned}\int_1^2 \frac{1}{x(x+1)}dx &= \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx \\ &= [\log|x| - \log|x+1|]_1^2 \\ &= (\log 2 - \log 3) - (-\log 2) = 2\log 2 - \log 3\end{aligned}$$

(12)
$$\begin{aligned}\int_2^3 \frac{1}{x^2-1}dx &= \int_2^3 \frac{1}{(x-1)(x+1)}dx \\ &= \frac{1}{2}\int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx \\ &= \frac{1}{2}[\log|x-1| - \log|x+1|]_1^2 \\ &= \frac{1}{2}(\log 2 - \log 4) - \frac{1}{2}(-\log 3) = -\frac{1}{2}\log 2 + \frac{1}{2}\log 3\end{aligned}$$

2.10.

(1) $\displaystyle\int_1^x \frac{t}{1+t^2}dt = \frac{1}{2}\left[\log(1+t^2)\right]_1^x = \frac{1}{2}\log(1+x^2) - \frac{1}{2}\log 2$

よって,

$$\frac{d}{dx}\left(\int_1^x \frac{t}{1+t^2}dt\right) = \frac{d}{dx}\left\{\frac{1}{2}\log(1+x^2) - \frac{1}{2}\log 2\right\} = \frac{x}{1+x^2}$$

(2) $\displaystyle\int e^{t^2}dt = F(t) + C$ とする．このとき,

$$\begin{aligned}\frac{d}{dx}\left(\int_2^{3x+1} e^{t^2}dt\right) &= \frac{d}{dx}[F(t)]_2^{3x+1} \\ &= \frac{d}{dx}\{F(3x+1) - F(2)\} = 3e^{(3x+1)^2}\end{aligned}$$

(3) $\displaystyle\int \frac{1}{t^3+1}dt = F(t) + C$ とする. このとき,

$$\begin{aligned}\frac{d}{dx}\left(\int_1^{x^2} \frac{1}{t^3+1}dt\right) &= \frac{d}{dx}\left[F(t)\right]_1^{x^2} \\ &= \frac{d}{dx}\left\{F(x^2) - F(1)\right\} = \frac{2x}{\sqrt{x^6+1}}\end{aligned}$$

(4) $\displaystyle\int_0^x (x-t)f(t)dt = x\int_0^x f(t)dt - \int_0^x tf(t)dt$
ここで, ある関数 $g(x)$ に対して,

$$\int g(t)dt = G(t) + C$$

とする. このとき,

$$\int_0^x g(t)dt = \left[G(t)\right]_0^x = G(x) - G(0)$$

より,

$$\frac{d}{dx}\left(\int_0^x g(t)dt\right) = \frac{d}{dx}\left(G(x) - G(0)\right) = G'(x) = g(x)$$

これより, $g(t) = f(t)$ または $g(t) = tf(t)$ として考えれば,

$$\begin{aligned}\frac{d}{dx}\left(\int_0^x (x-t)f(t)dt\right) &= \frac{d}{dx}\left(x\int_0^x f(t)dt - \int_0^x tf(t)dt\right) \\ &= \int_0^x f(t)dt + xf(x) - xf(x) \\ &= \int_0^x f(t)dt\end{aligned}$$

2.11. 定理 1.30 (ロピタルの定理) を使えばよい.

(1) $\displaystyle\lim_{x\to 0}\frac{\int_0^x \log(\cos t)dt}{x} = \lim_{x\to 0}\log(\cos x) = 0$

(2) $\displaystyle\lim_{x\to 0}\frac{1}{\sin x}\int_0^{2x}\frac{1}{\sqrt{t^3+1}}dt = \lim_{x\to 0}\frac{1}{\cos x}\frac{2}{\sqrt{8x^3+1}} = 2$

(3) 最初に $\displaystyle\frac{d}{dx}\left(\int_x^{2x} e^{-t^2}dt\right)$ を計算しよう. $\displaystyle\int e^{-t^2}dt = F(t) + C$ とすると,

$$\begin{aligned}\frac{d}{dx}\left(\int_x^{2x} e^{-t^2}dt\right) &= \frac{d}{dx}\left[F(t)\right]_x^{2x} \\ &= \frac{d}{dx}\left(F(2x) - F(x)\right) = 2e^{-4x^2} - e^{-x^2}\end{aligned}$$

よって, 定理 1.30（ロピタルの定理）から

$$\lim_{x\to 0}\frac{1}{x}\int_x^{2x} e^{-t^2}dt = \lim_{x\to 0}\left(2e^{-4x^2} - e^{-x^2}\right) = 1$$

2.12

(1) $\displaystyle\int_1^e \log x dx = [x\log x]_1^e - \int_1^e dx = e - [x]_1^e = e - e + 1 = 1$

(2) $\displaystyle\int_0^\pi x\cos x dx = [x\sin x]_0^\pi - \int_0^\pi \sin x dx = -[\cos x]_0^\pi = -1 - 1 = -2$

(3)
$$\begin{aligned}\int_0^1 xa^x dx &= \left[x\frac{a^x}{\log a}\right]_0^1 - \int_0^1 \frac{a^x}{\log a}dx \\ &= \frac{a}{\log a} - \left[\frac{a^x}{(\log a)^2}\right]_0^1 \\ &= \frac{a}{\log a} - \frac{a}{(\log a)^2} + \frac{1}{(\log a)^2} = \frac{a\log a - a + 1}{(\log a)^2}\end{aligned}$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x dx = [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx = [\sin x]_0^{\frac{\pi}{2}} = 1$

(5)
$$\begin{aligned}\int_0^e x\log x dx &= \left[\frac{1}{2}x^2\log x\right]_0^e - \int_0^e \frac{1}{2}x dx \\ &= \frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_0^e = \frac{1}{2}e^2 - \frac{1}{4}e^2 = \frac{1}{4}e^2\end{aligned}$$

(6) $\displaystyle\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = e - [e^x]_0^1 = e - e + 1 = 1$

(7)
$$\begin{aligned}\int_0^1 \tan^{-1} x dx &= [x\tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2}dx \\ &= \frac{\pi}{4} - \left[\frac{1}{2}\log(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\log 2\end{aligned}$$

(8)
$$\begin{aligned}\int_1^e (\log x)^2 dx &= \left[x(\log x)^2\right]_1^e - 2\int_1^e x(\log x)\cdot\frac{1}{x}dx \\ &= e - 2\int_1^e \log x dx \\ &= e - 2\left([x\log x]_1^e - \int_1^e dx\right) \\ &= e - 2\left(e - [x]_1^e\right) = e - 2(e - e + 1) = e - 2\end{aligned}$$

(9) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin x \cdot \sin^2 x dx$

$$= [-\cos x \sin^2 x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot 2\sin x \cos x dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin x(1-\sin^2 x)dx = 2\int_0^{\frac{\pi}{2}} \sin x dx - 2\int_0^{\frac{\pi}{2}} \sin^3 x dx$$

よって, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x dx = \frac{2}{3}[-\cos x]_0^{\frac{\pi}{2}} = \frac{2}{3}$

(10) $\displaystyle\int_1^e x^3 \log x dx = \left[\frac{1}{4}x^4 \log x\right]_1^e - \int_1^e \frac{1}{4}x^3 dx$

$$= \frac{1}{4}e^4 - \left[\frac{1}{16}x^4\right]_1^e$$

$$= \frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16} = \frac{3}{16}e^4 + \frac{1}{16}$$

(11) $\displaystyle\int_0^1 x\sqrt{x+1}dx = \left[\frac{2}{3}x(x+1)^{\frac{3}{2}}\right]_0^1 - \int_0^1 \frac{2}{3}(x+1)^{\frac{3}{2}}dx$

$$= \frac{2}{3}\cdot 2^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}\left[(x+1)^{\frac{5}{2}}\right]_0^1$$

$$= \frac{1}{3}\cdot 2^{\frac{5}{2}} - \frac{4}{15}\cdot 2^{\frac{5}{2}} + \frac{4}{15} = \frac{1}{15}2^{\frac{5}{2}} + \frac{4}{15}$$

(12) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x dx = [x\sin^{-1} x]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}}dx$

ここで, $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}}dx$ において $t = 1-x^2$ とおく. $-2xdx = dt$,

x	$0 \to \frac{1}{\sqrt{2}}$
t	$1 \to \frac{1}{2}$

. よって,

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int_1^{\frac{1}{2}} \frac{1}{\sqrt{t}}dt = \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{t}}dt = [\sqrt{t}]_{\frac{1}{2}}^1 = \left(1-\frac{1}{\sqrt{2}}\right)$$

ゆえに,

$$\int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x dx = [x\sin^{-1} x]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}}dx$$

$$= \frac{\pi}{4\sqrt{2}} - \left(1-\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

2.13

(1) $1-x=t$ とおくと, $-dx=dt$,

x	$0 \to 1$
t	$1 \to 0$

. よって,

$$\int_0^1 \sqrt{1-x}dx = \int_1^0 \sqrt{t}\cdot(-1)dt = \int_0^1 \sqrt{t}dt = \left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

(2) $2x=t$ とおくと, $2dx=dt$,

x	$0 \to \pi$
t	$0 \to 2\pi$

. よって,

$$\int_0^{\pi} \sin 2x dx = \frac{1}{2}\int_0^{2\pi} \sin t dt = \frac{1}{2}[-\cos t]_0^{2\pi} = 0$$

(3) $2x=t$ とおくと, $2dx=dt$,

x	$-1 \to 1$
t	$-2 \to 2$

. よって,

$$\int_{-1}^{1} e^{2x}dx = \frac{1}{2}\int_{-2}^{2} e^t dt = \frac{1}{2}[e^t]_{-2}^2 = \frac{e^4-1}{2e^2}$$

(4) $\displaystyle\int_1^2 \frac{1}{4x^2-1}dx = \int_1^2 \frac{1}{(2x-1)(2x+1)}dx.$
ここで,

$$\frac{1}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1}$$

とおく. 両辺を $(2x-1)(2x+1)$ 倍すると

$$1 = A(2x+1) + B(2x-1)$$

ここで, 両辺に $x=\frac{1}{2}$ を代入すると, $A=\frac{1}{2}$, $x=-\frac{1}{2}$ を代入すると, $B=-\frac{1}{2}$ を得る. よって,

$$\begin{aligned}\int_1^2 \frac{1}{4x^2-1}dx &= \int_1^2 \frac{1}{(2x-1)(2x+1)}dx \\ &= \frac{1}{2}\int_1^2 \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)dx \\ &= \frac{1}{2}\left[\frac{1}{2}\log|2x-1| - \frac{1}{2}\log|2x+1|\right]_1^2 dx \\ &= \frac{1}{4}(\log 3 - \log 5) - \frac{1}{4}(\log 1 - \log 3) \\ &= \frac{1}{2}\log 3 - \frac{1}{4}\log 5\end{aligned}$$

(5) $\displaystyle\int_0^1 \frac{2x+1}{x^2+3x+2}dx = \int_0^1 \frac{2x+1}{(x+1)(x+2)}dx$
ここで,

$$\frac{2x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

とおく. 両辺を $(x+1)(x+2)$ 倍すると

$$2x+1 = A(x+2) + B(x+1)$$

ここで, 両辺に $x=-1$ を代入すると, $A=-1$, $x=-2$ を代入すると, $B=3$ を得る. よって,

$$\begin{aligned}\int_0^1 \frac{2x+1}{x^2+3x+2}dx &= \int_0^1 \frac{2x+1}{(x+1)(x+2)}dx \\ &= \int_0^1 \left(\frac{3}{x+2} - \frac{1}{x+1}\right)dx \\ &= [3\log|x+2| - \log|x+1|]_0^1 \\ &= (3\log 3 - \log 2) - (3\log 2 - \log 1) \\ &= 3\log 3 - 4\log 2\end{aligned}$$

(6) $$\begin{aligned}\int_0^3 (5x+2)\sqrt{x+1}dx &= \int_0^3 (5x+2)(x+1)^{\frac{1}{2}}dx \\ &= \left[\frac{2}{3}(5x+2)(x+1)^{\frac{3}{2}}\right]_0^3 - \frac{2}{3}\int_0^3 5\cdot(x+1)^{\frac{3}{2}}dx \\ &= \frac{2}{3}\left(17\cdot 4^{\frac{3}{2}} - 2\cdot 1\right) - \frac{10}{3}\left[\frac{2}{5}(x+1)^{\frac{5}{2}}\right]_0^3 \\ &= \frac{2}{3}(17\cdot 8 - 2) - \frac{4}{3}\left(4^{\frac{5}{2}} - 1\right) = 48\end{aligned}$$

(7) $$\begin{aligned}\int_{-1}^{\frac{1}{2}} x(2x+3)^{\frac{3}{2}}dx &= \left[\frac{1}{5}x(2x+3)^{\frac{5}{2}}\right]_{-1}^{\frac{1}{2}} - \int_{-1}^{\frac{1}{2}} \frac{1}{5}(2x+3)^{\frac{5}{2}}dx \\ &= \frac{1}{5}\left\{\frac{1}{2}\cdot 4^{\frac{5}{2}} - (-1)\right\} - \frac{1}{5}\cdot\frac{1}{7}\left[(2x+3)^{\frac{7}{2}}\right]_{-1}^{\frac{1}{2}} \\ &= \frac{17}{5} - \frac{1}{35}(4^{\frac{7}{2}} - 1) = -\frac{8}{35}\end{aligned}$$

(8) $1-x=t$ とおくと, $-dx=dt$,

x	$0\to 1$
t	$1\to 0$

. よって,

$$\int_0^1 \sqrt[3]{1-x}dx = \int_1^0 t^{\frac{1}{3}}\cdot(-1)dt = \int_0^1 t^{\frac{1}{3}}dt = \frac{3}{4}[t^{\frac{4}{3}}]_0^1 = \frac{3}{4}$$

(9) $\displaystyle\int_{-1}^{1} e^{-x}dx$ において $-x=t$ とおく．$-dx=dt$,

x	$-1 \to 1$
t	$1 \to -1$

. よって,

$$\int_{-1}^{1} e^{-x}dx = \int_{1}^{-1} e^{t}\cdot(-1)dt = \int_{-1}^{1} e^{t}dt$$

これより,

$$\int_{-1}^{1}(e^{x}-e^{-x})dx = \int_{-1}^{1} e^{x}dx - \int_{-1}^{1} e^{-x}dx = \int_{-1}^{1} e^{x}dx - \int_{-1}^{1} e^{t}dt = 0$$

(10) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2xdx$ において, $2x=t$ とおく．$2dx=dt$,

x	$-\dfrac{\pi}{2} \to \dfrac{\pi}{2}$
t	$-\pi \to \pi$

. よって,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2xdx &= \frac{1}{2}\int_{-\pi}^{\pi}\sin tdt \\ &= \frac{1}{2}[-\cos t]_{-\pi}^{\pi} \\ &= \frac{-1}{2}\{\cos\pi - \cos(-\pi)\} \\ &= \frac{-1}{2}\{-1-(-1)\} = 0\end{aligned}$$

一方, $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 3xdx$ において $3x=s$ とおく．$3dx=ds$,

x	$-\dfrac{\pi}{2} \to \dfrac{\pi}{2}$
s	$-\dfrac{3}{2}\pi \to \dfrac{3}{2}\pi$

.

よって,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 3xdx &= \frac{1}{3}\int_{-\frac{3}{2}\pi}^{\frac{3}{2}\pi}\cos sds \\ &= \frac{1}{3}[\sin s]_{-\frac{3}{2}\pi}^{\frac{3}{2}\pi} \\ &= \frac{1}{3}(-1-1) = -\frac{2}{3}\end{aligned}$$

ゆえに, $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin 2x - \cos 3x)dx = 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$

(11)
$$\begin{aligned}\int_{-1}^{1}\frac{1-x}{1+x^2}dx &= \int_{-1}^{1}\left(\frac{1}{1+x^2}-\frac{x}{1+x^2}\right)dx \\ &= \left[\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_{-1}^{1} \\ &= \left(\frac{\pi}{4}-\frac{1}{2}\log 2\right) - \left(-\frac{\pi}{4}-\frac{1}{2}\log 2\right) = \frac{\pi}{2}\end{aligned}$$

(12) $\displaystyle\int_{-2}^{2} x\sqrt{9x^2-4}dx = \int_{-2}^{0} x\sqrt{9x^2-4}dx + \int_{0}^{2} x\sqrt{9x^2-4}dx.$

ここで, $\displaystyle\int_{-2}^{0} x\sqrt{9x^2-4}dx$ において, $-x=t$ とおく. $-dx=dt$,

x	$-2 \to 0$
t	$2 \to 0$

.

よって,

$$\int_{-2}^{0} x\sqrt{9x^2-4}dx = -\int_{2}^{0}(-t)\sqrt{9t^2-4}dt = -\int_{0}^{2} t\sqrt{9t^2-4}dt$$

ゆえに,

$$\begin{aligned}\int_{-2}^{2} x\sqrt{9x^2-4}dx &= \int_{-2}^{0} x\sqrt{9x^2-4}dx + \int_{0}^{2} x\sqrt{9x^2-4}dx \\ &= -\int_{0}^{2} t\sqrt{9t^2-4}dt + \int_{0}^{2} x\sqrt{9x^2-4}dx = 0\end{aligned}$$

(13) $x^2=t$ とおくと, $2xdx=dt$,

x	$0 \to 1$
t	$0 \to 1$

. よって,

$$\int_{0}^{1} xe^{x^2}dx = \frac{1}{2}\int_{0}^{1} e^t dt = \frac{1}{2}[e^t]_0^1 = \frac{1}{2}(e-1)$$

(14) $1+x^2=t$ とおくと, $2xdx=dt$,

x	$0 \to 1$
t	$1 \to 2$

. よって,

$$\int_{0}^{1} x\sqrt{1+x^2}dx = \frac{1}{2}\int_{1}^{2}\sqrt{t}dt = \frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^2 = \frac{1}{3}(2\sqrt{2}-1)$$

(15)
$$\begin{aligned}\int_{0}^{2}\frac{1}{\sqrt{x+2}+\sqrt{x}}dx &= \frac{1}{2}\int_{0}^{2}(\sqrt{x+2}-\sqrt{x})dx \\ &= \frac{1}{2}\left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^2 \\ &= \frac{1}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \frac{1}{3}(8-4\sqrt{2})\end{aligned}$$

(16) $2x+1=t$ とおくと, $2dx=dt$,

x	$1 \to 2$
t	$3 \to 5$

. よって,

$$\begin{aligned}\int_{1}^{2}\frac{1}{\sqrt[5]{2x+1}}dx &= \frac{1}{2}\int_{3}^{5} t^{-\frac{1}{5}}dt \\ &= \frac{1}{2}\left[\frac{5}{4}t^{\frac{4}{5}}\right]_3^5 = \frac{5}{8}\left(5^{\frac{4}{5}} - 3^{\frac{4}{5}}\right)\end{aligned}$$

(17) $4-x^2=t$ とおくと, $-2xdx=dt$,

x	$1\to 2$
t	$3\to 0$

. よって,

$$\int_1^2 x\sqrt{4-x^2}dx=-\frac{1}{2}\int_3^0\sqrt{t}dt=\frac{1}{2}\int_0^3 t^{\frac{1}{2}}dt=\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^3=\sqrt{3}$$

(18) $\sqrt{x}=t$ とおくと, $x=t^2$ より $dx=2tdt$,

x	$0\to\frac{\pi^2}{4}$
t	$0\to\frac{\pi}{2}$

. よって,

$$\int_0^{\frac{\pi^2}{4}}\sin\sqrt{x}dx=\int_0^{\frac{\pi}{2}}2t\sin tdt=[-2t\cos t]_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\cos tdt=2[\sin t]_0^{\frac{\pi}{2}}=2$$

2.14.

(1) $\displaystyle\int_0^{2\pi}\sin^6 xdx=\int_0^{\frac{\pi}{2}}\sin^6 xdx+\int_{\frac{\pi}{2}}^{\pi}\sin^6 xdx+\int_{\pi}^{\frac{3}{2}\pi}\sin^6 xdx+\int_{\frac{3}{2}\pi}^{2\pi}\sin^6 xdx.$

例えば, $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\sin^6 xdx$ において, $t=x-\dfrac{\pi}{2}$ とおくと, $dx=dt$,

x	$\frac{\pi}{2}\to\pi$
t	$0\to\frac{\pi}{2}$

.

よって,

$$\int_{\frac{\pi}{2}}^{\pi}\sin^6 xdx=\int_0^{\frac{\pi}{2}}\sin^6\left(t+\frac{\pi}{2}\right)dt=\int_0^{\frac{\pi}{2}}\cos^6 tdt=\int_0^{\frac{\pi}{2}}\sin^6 xdx$$

同様に $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\sin^6 xdx=\int_{\pi}^{\frac{3}{2}\pi}\sin^6 xdx=\int_{\frac{3}{2}\pi}^{2\pi}\sin^6 xdx=\int_0^{\frac{\pi}{2}}\sin^6 xdx$ を得る.

よって, $\displaystyle\int_0^{2\pi}\sin^6 xdx=4\int_0^{\frac{\pi}{2}}\sin^6 xdx=4\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}=\frac{5}{8}\pi.$

(2) $\displaystyle\int_{-\frac{3}{2}\pi}^{\pi}\cos^3 xdx=\int_{-\frac{3}{2}\pi}^{0}\cos^3 xdx+\int_0^{\pi}\cos^3 xdx.$ ここで, $\displaystyle\int_{-\frac{3}{2}\pi}^{0}\cos^3 xdx$ において $x=-t$ とおくと, $dx=-dt$,

x	$-\frac{3}{2}\pi\to 0$
t	$\frac{3}{2}\pi\to 0$

. よって,

$$\int_{-\frac{3}{2}\pi}^{0}\cos^3 xdx=-\int_{\frac{3}{2}\pi}^{0}\cos^3(-t)dt=\int_0^{\frac{3}{2}\pi}\cos^3 tdt$$

これより,

$$\int_{-\frac{3}{2}\pi}^{\pi}\cos^3 xdx=2\int_0^{\frac{\pi}{2}}\cos^3 xdx+2\int_{\frac{\pi}{2}}^{\pi}\cos^3 xdx+\int_{\pi}^{\frac{3}{2}\pi}\cos^3 xdx$$

ここで, $\int_{\frac{\pi}{2}}^{\pi}\cos^3 xdx$ において $t = x - \dfrac{\pi}{2}$ とおくと, $dt = dx$,

x	$\dfrac{\pi}{2} \to \pi$
t	$0 \to \dfrac{\pi}{2}$

.

よって,

$$\int_{\frac{\pi}{2}}^{\pi}\cos^3 xdx = \int_0^{\frac{\pi}{2}}\cos^3\left(t+\frac{\pi}{2}\right)dt = -\int_0^{\frac{\pi}{2}}\sin^3 tdt = -\int_0^{\frac{\pi}{2}}\cos^3 tdt$$

一方, $\int_{\pi}^{\frac{3}{2}\pi}\cos^3 xdx$ において $s = x - \pi$ とおくと, $ds = dx$,

x	$\pi \to \dfrac{3}{2}\pi$
s	$0 \to \dfrac{\pi}{2}$

.

よって,

$$\int_{\pi}^{\frac{3}{2}\pi}\cos^3 xdx = \int_0^{\frac{\pi}{2}}\cos^3(s+\pi)dx = -\int_0^{\frac{\pi}{2}}\cos^3 sds$$

ゆえに,

$$\begin{aligned}\int_{-\frac{3}{2}\pi}^{\pi}\cos^3 xdx &= 2\int_0^{\frac{\pi}{2}}\cos^3 xdx + 2\int_{\frac{\pi}{2}}^{\pi}\cos^3 xdx + \int_{\pi}^{\frac{3}{2}\pi}\cos^3 xdx\\ &= 2\int_0^{\frac{\pi}{2}}\cos^3 xdx - 3\int_0^{\frac{\pi}{2}}\cos^3 xdx\\ &= -\int_0^{\frac{\pi}{2}}\cos^3 xdx = -\frac{2}{3}\end{aligned}$$

(3) $\int_{-\pi}^{\pi}\sin^5 2xdx = \int_{-\pi}^{0}\sin^5 2xdx + \int_0^{\pi}\sin^5 2xdx$. ここで, $\int_{-\pi}^{0}\sin^5 2xdx$ において, $x = -t$ とおくと, $dx = -dt$,

x	$-\pi \to 0$
t	$\pi \to 0$

. よって,

$$\begin{aligned}\int_{-\pi}^{\pi}\sin^5 2xdx &= \int_{-\pi}^{0}\sin^5 2xdx + \int_0^{\pi}\sin^5 2xdx\\ &= \int_{\pi}^{0}\sin^5 2(-t)(-1)dt + \int_0^{\pi}\sin^5 2xdx\\ &= -\int_0^{\pi}\sin^5 2tdt + \int_0^{\pi}\sin^5 2xdx = 0\end{aligned}$$

(4) 例題 2.15 と同様の方法から $\int_0^{\frac{\pi}{2}}\dfrac{\sin^2 x\cos x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}}\dfrac{\cos^2 x\sin x}{\cos x+\sin x}dx$ を得る.

よって,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\cos x}{\sin x+\cos x}dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\cos x}{\sin x+\cos x}dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\sin x}{\cos x+\sin x}dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x(\sin x+\cos x)}{\sin x+\cos x}dx \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin 2x dx \\ &= \frac{1}{8}\left[-\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\end{aligned}$$

2.15.

(1) $t=\pi-x$ とおくと, $dt=-dx$,

x	$0\to\pi$
t	$\pi\to 0$

. よって,

$$\begin{aligned}\int_0^{\pi} x\sin^2 x dx &= -\int_{\pi}^{0}(\pi-t)\sin^2(\pi-t)dt \\ &= \int_0^{\pi}(\pi-t)\sin^2 t dt \\ &= \pi\int_0^{\pi}\sin^2 t dt - \int_0^{\pi} t\sin^2 t dt\end{aligned}$$

ゆえに

$$\begin{aligned}\int_0^{\pi} x\sin^2 x dx &= \frac{\pi}{2}\int_0^{\pi}\sin^2 t dt \\ &= \frac{\pi}{2}\int_0^{\pi}\frac{1-\cos 2t}{2}dt \\ &= \frac{\pi}{4}\left[t-\frac{1}{2}\sin 2t\right]_0^{\pi} = \frac{\pi^2}{4}\end{aligned}$$

(2) $x=\tan t$ とおくと, $dx=\dfrac{1}{\cos^2 t}dt=(1+\tan^2 t)dt$,

x	$0\to 1$
t	$0\to\dfrac{\pi}{4}$

. よって,

$$\begin{aligned}\int_0^1 \frac{\log(1+x)}{1+x^2}dx &= \int_0^{\frac{\pi}{4}}\log(1+\tan t)dt \\ &= \int_0^{\frac{\pi}{4}}\log\left(\frac{\cos t+\sin t}{\cos t}\right)dt \\ &= \int_0^{\frac{\pi}{4}}\log\frac{\sqrt{2}\cos(t-\frac{\pi}{4})}{\cos t}dt \\ &= \int_0^{\frac{\pi}{4}}\log\sqrt{2}dt + \int_0^{\frac{\pi}{4}}\log\cos\left(t-\frac{\pi}{4}\right)dt - \int_0^{\frac{\pi}{4}}\log\cos t dt\end{aligned}$$

ここで, $\int_0^{\frac{\pi}{4}} \log\cos\left(t-\frac{\pi}{4}\right)dt$ において $t-\frac{\pi}{4}=-s$ とおくと, $dt=-ds$,

t	$0\to\frac{\pi}{4}$
s	$\frac{\pi}{4}\to 0$

. よって,

$$\int_0^{\frac{\pi}{4}} \log\cos\left(t-\frac{\pi}{4}\right)dt = \int_{\frac{\pi}{4}}^{0} \log\cos(-s)(-1)ds = \int_0^{\frac{\pi}{4}} \log\cos s ds$$

ゆえに,

$$\begin{aligned}(与式) &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}dt + \int_0^{\frac{\pi}{4}} \log\cos\left(t-\frac{\pi}{4}\right)dt - \int_0^{\frac{\pi}{4}} \log\cos t dt \\ &= (\log\sqrt{2})[x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \log\cos s ds - \int_0^{\frac{\pi}{4}} \log\cos t dt \\ &= \frac{\pi}{8}\log 2\end{aligned}$$

2.16.

(1) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{\frac{n}{n}}\right)=\int_0^1 e^x dx=[e^x]_0^1=e-1$

(2) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2}{n}\pi+\cdots+\sin\frac{n}{n}\pi\right)=\int_0^1 \sin\pi x dx$

$$=\left[-\frac{1}{\pi}\cos\pi x\right]_0^1=\frac{2}{\pi}$$

(3) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+\cdots+\frac{1}{\sqrt{2n-1}}\right)$

$$\begin{aligned}&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt{n}}{\sqrt{n}}+\frac{\sqrt{n}}{\sqrt{n+1}}+\cdots+\frac{\sqrt{n}}{\sqrt{2n-1}}\right)\\ &=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{0}{n}}}+\frac{1}{\sqrt{1+\frac{1}{n}}}+\cdots+\frac{1}{\sqrt{1+\frac{n-1}{n}}}\right)\\ &=\int_0^1\frac{1}{\sqrt{1+x}}dx=[2\sqrt{1+x}]_0^1=2\sqrt{2}-2\end{aligned}$$

(4) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}\log\frac{n+1}{n}+\frac{1}{n+2}\log\frac{n+2}{n}+\cdots+\frac{1}{n+n}\log\frac{n+n}{n}\right)$

$$\begin{aligned}&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{n+1}\log\frac{n+1}{n}+\frac{n}{n+2}\log\frac{n+2}{n}+\cdots+\frac{n}{n+n}\log\frac{n+n}{n}\right)\\ &=\lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{1+\frac{1}{n}}\log\left(1+\frac{1}{n}\right)+\frac{1}{1+\frac{2}{n}}\log\left(1+\frac{2}{n}\right)+\cdots+\frac{1}{1+\frac{n}{n}}\log\left(1+\frac{n}{n}\right)\right\}\\ &=\int_0^1\frac{\log(1+x)}{1+x}dx\end{aligned}$$

ここで, $\log(1+x)=t$ とおくと, $\frac{1}{1+x}dx=dt$,

x	$0\to 1$
t	$0\to \log 2$

. よって,

$$与式 =\int_0^{\log 2} tdt=\left[\frac{1}{2}t^2\right]_0^{\log 2}=\frac{1}{2}(\log 2)^2$$

2.17.

(1) $0<x\leqq\frac{\pi}{4}$ のとき, $1<\frac{1}{\sqrt{1-\sin x}}<\frac{1}{\sqrt{1-x}}$ より,

$$\int_0^{\frac{\pi}{4}}dx<\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{1-\sin x}}dx<\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{1-x}}dx$$

である. ここで,

$$\int_0^{\frac{\pi}{4}}dx=[x]_0^{\frac{\pi}{4}}=\frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{1-x}}dx=[-2\sqrt{1-x}]_0^{\frac{\pi}{4}}=2-\sqrt{4-\pi}$$

よって,

$$\frac{\pi}{4}<\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{1-\sin x}}dx<2-\sqrt{4-\pi}$$

(2) $0<x<\frac{1}{2}$ のとき, $\sqrt{1-x^2}<\sqrt{1-x^4}<1$ より,

$$\int_0^{\frac{1}{2}}dx<\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^4}}dx<\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}dx$$

ここで,

$$\int_0^{\frac{1}{2}}dx=[x]_0^{\frac{1}{2}}=\frac{1}{2}$$

$$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}dx=[\sin^{-1}x]_0^{\frac{1}{2}}=\frac{\pi}{6}$$

よって,

$$\frac{1}{2}<\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^4}}dx<\frac{\pi}{6}$$

(3) $0\leqq x\leqq 1$ のとき, $1-x^2\leqq 1-x^4\leqq 2(1-x^2)$ より,

$$\sqrt{1-x^2}\leqq\sqrt{1-x^4}\leqq\sqrt{2(1-x^2)}$$

これより,

$$\int_0^1\sqrt{1-x^2}dx\leqq\int_0^1\sqrt{1-x^4}dx\leqq\int_0^1\sqrt{2(1-x^2)}dx$$

ここで, $\int_0^1 \sqrt{1-x^2}dx$ において, $x=\sin t$ とおくと $dx=\cos t dt$,

x	$0 \to 1$
t	$0 \to \frac{\pi}{2}$

より,

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \cos^2 t dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt$$
$$= \left[\frac{t}{2}+\frac{1}{4}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

よって,

$$\frac{\pi}{4} \leqq \int_0^1 \sqrt{1-x^4}dx \leqq \frac{\sqrt{2}}{4}\pi$$

(4) $0 \leqq x \leqq 1$ のとき, $0 \leqq x^2 \leqq x$ である. これより

$$e^{-\frac{x}{2}} \leqq e^{-\frac{x^2}{2}} \leqq 1$$

よって,

$$\int_0^1 e^{-\frac{x}{2}}dx \leqq \int_0^1 e^{-\frac{x^2}{2}}dx \leqq \int_0^1 dx$$

ここで,

$$\int_0^1 e^{-\frac{x}{2}}dx = \left[-2e^{-\frac{x}{2}}\right]_0^1 = 2\left(1-\frac{1}{\sqrt{e}}\right)$$
$$\int_0^1 dx = [x]_0^1 = 1$$

ゆえに,

$$2\left(1-\frac{1}{\sqrt{e}}\right) \leqq \int_0^1 e^{-\frac{x^2}{2}}dx \leqq 1$$

2.18.

(1) $0 \leqq x \leqq 1$ のとき, $1 \leqq x^3+1 \leqq x^2+1$ より

$$\int_0^1 \frac{1}{x^2+1}dx \leqq \int_0^1 \frac{1}{x^3+1}dx \leqq \int_0^1 dx$$

ここで,

$$\int_0^1 \frac{1}{x^2+1}dx = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$$
$$\int_0^1 dx = [x]_0^1 = 1$$

よって,

$$\frac{\pi}{4} \leqq \int_0^1 \frac{1}{x^3+1}dx \leqq 1$$

なお, 部分分数分解を利用すれば次のように定積分の計算ができる.

$$\int_0^1 \frac{1}{x^3+1}dx$$
$$= \int_0^1 \left(\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)} \right) dx$$
$$= \int_0^1 \left(\frac{1}{3(x+1)} - \frac{x-\frac{1}{2}}{3\{(x-\frac{1}{2})^2+\frac{3}{4}\}} + \frac{1}{2\{(x-\frac{1}{2})^2+\frac{3}{4}\}} \right) dx$$
$$= \frac{1}{3}[\log(x+1)]_0^1 - \frac{1}{6}[\log(x^2-x+1)]_0^1 + \frac{1}{2}\left[\sqrt{\frac{4}{3}} \tan^{-1}\sqrt{\frac{4}{3}}\left(x-\frac{1}{2}\right) \right]_0^1$$
$$= \frac{1}{3}\log 2 + \frac{\sqrt{3}}{9}\pi$$

(2) $k < x < k+1$ のとき, $\frac{1}{(k+1)^2} < \frac{1}{x^2} < \frac{1}{k^2}$ である. これより,

$$\int_k^{k+1} \frac{1}{(k+1)^2}dx < \int_k^{k+1} \frac{1}{x^2}dx < \int_k^{k+1} \frac{1}{k^2}dx$$
$$\Longleftrightarrow \frac{1}{(k+1)^2} < \int_k^{k+1} \frac{1}{x^2}dx < \frac{1}{k^2}$$

よって, $k = 1, 2, ..., n$ について足し合わせると

$$\sum_{k=1}^{n} \frac{1}{(k+1)^2} < \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x^2}dx < \sum_{k=1}^{n} \frac{1}{k^2}$$

ここで,

$$\sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x^2}dx = \int_1^{n+1} \frac{1}{x^2}dx = \left[-\frac{1}{x} \right]_1^{n+1} = 1 - \frac{1}{n+1}$$

よって,

$$\sum_{k=1}^{n} \frac{1}{(k+1)^2} < 1 - \frac{1}{n+1} < \sum_{k=1}^{n} \frac{1}{k^2}$$

(3) $k < x < k+1$ のとき, $\frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{k}}$ である. これより,

$$\int_k^{k+1} \frac{1}{\sqrt{k+1}}dx < \int_k^{k+1} \frac{1}{\sqrt{x}}dx < \int_k^{k+1} \frac{1}{\sqrt{k}}dx$$
$$\Longleftrightarrow \frac{1}{\sqrt{k+1}} < \int_k^{k+1} \frac{1}{\sqrt{x}}dx < \frac{1}{\sqrt{k}}$$

よって, $k = 1, 2, ..., n$ について足し合わせると

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k+1}} < \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{\sqrt{x}}dx < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

ここで,

$$\sum_{k=1}^{n}\int_{k}^{k+1}\frac{1}{\sqrt{x}}dx=\int_{1}^{n+1}\frac{1}{\sqrt{x}}dx=[2\sqrt{x}]_{1}^{n+1}=2(\sqrt{n+1}-1)$$

よって,

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}}<2(\sqrt{n+1}-1)<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$

(4) $k-1<x<k$ のとき, $\dfrac{1}{2k+1}<\dfrac{1}{2x+1}<\dfrac{1}{2k-1}$ である. これより,

$$\int_{k-1}^{k}\frac{1}{2k+1}dx<\int_{k-1}^{k}\frac{1}{2x+1}dx<\int_{k-1}^{k}\frac{1}{2k-1}dx$$

$$\Longleftrightarrow\frac{1}{2k+1}<\int_{k-1}^{k}\frac{1}{2x+1}dx<\frac{1}{2k-1}$$

よって, $k=1,2,...,n$ について足し合わせると

$$\sum_{k=1}^{n}\frac{1}{2k+1}<\sum_{k=1}^{n}\int_{k-1}^{k}\frac{1}{2x+1}dx<\sum_{k=1}^{n}\frac{1}{2k-1}$$

ここで,

$$\sum_{k=1}^{n}\int_{k-1}^{k}\frac{1}{2x+1}dx=\int_{0}^{n}\frac{1}{2x+1}dx=\left[\frac{1}{2}\log(2x+1)\right]_{0}^{n}=\frac{1}{2}\log(2n+1)$$

よって,

$$\sum_{k=1}^{n}\frac{1}{2k+1}<\frac{1}{2}\log(2n+1)<\sum_{k=1}^{n}\frac{1}{2k-1}$$

2.19.

(1) $\displaystyle\int_{0}^{\infty}e^{-x}dx=\lim_{M\to\infty}\int_{0}^{M}e^{-x}dx=\lim_{M\to\infty}\left[-e^{-x}\right]_{0}^{M}=\lim_{M\to\infty}\left(1-e^{-M}\right)=1$

(2) $\displaystyle\int_{1}^{\infty}\frac{1}{\sqrt{1+x}}dx=\lim_{M\to\infty}\int_{1}^{M}\frac{1}{\sqrt{1+x}}dx=\lim_{M\to\infty}\left[2\sqrt{1+x}\right]_{0}^{M}$

$$=\lim_{M\to\infty}\left(2\sqrt{1+M}-2\right)=+\infty$$

よって, 広義積分は存在しない.

(3) $\displaystyle\int_{2}^{\infty}\frac{1}{(1-x)^2}dx=\lim_{M\to\infty}\int_{2}^{M}\frac{1}{(1-x)^2}dx=\lim_{M\to\infty}\left[\frac{1}{1-x}\right]_{2}^{M}$

$$=\lim_{M\to\infty}\left(\frac{1}{1-M}+1\right)=1$$

(4) $$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{M\to\infty}\int_0^M \frac{1}{1+x^2}dx = \lim_{M\to\infty}\left[\tan^{-1}x\right]_0^M = \lim_{M\to\infty}\tan^{-1}M = \frac{\pi}{2}$$

(5) $$\int_1^{\infty} \frac{1}{(1-x)^2}dx = \lim_{\substack{M\to\infty\\ N\to1+0}}\int_N^M \frac{1}{(1-x)^2}dx = \lim_{\substack{M\to\infty\\ N\to1+0}}\left[\frac{1}{1-x}\right]_N^M = \lim_{\substack{M\to\infty\\ N\to1+0}}\left(\frac{1}{1-M}-\frac{1}{1-N}\right) = +\infty$$

よって, 広義積分は存在しない.

(6) $$\int_2^{\infty} \frac{1}{x(x-1)}dx = \lim_{M\to\infty}\int_2^M \frac{1}{x(x-1)}dx = \lim_{M\to\infty}\int_2^M\left(\frac{1}{x-1}-\frac{1}{x}\right)dx = \lim_{M\to\infty}\left[\log\frac{x-1}{x}\right]_2^M = \lim_{M\to\infty}\left(\log\frac{M-1}{M}-\log\frac{1}{2}\right) = \log 2$$

(7) $$\int_1^{2} \frac{1}{(x-1)(x-2)}dx = \lim_{\substack{M\to2-0\\ N\to1+0}}\int_N^M \frac{1}{(x-1)(x-2)}dx = \lim_{\substack{M\to2-0\\ N\to1+0}}\int_N^M\left(\frac{1}{x-2}-\frac{1}{x-1}\right)dx = \lim_{\substack{M\to2-0\\ N\to1+0}}\left[\log\left|\frac{x-2}{x-1}\right|\right]_N^M = \lim_{\substack{M\to2-0\\ N\to1+0}}\left(\log\left|\frac{M-2}{M-1}\right|-\log\left|\frac{N-2}{N-1}\right|\right) = -\infty$$

よって, 広義積分は存在しない.

(8) $$\int_{-2}^{0} \frac{1}{\sqrt{-x^2-2x}}dx = \lim_{\substack{M\to0-\\ N\to-2+0}}\int_N^M \frac{1}{\sqrt{-x^2-2x}}dx = \lim_{\substack{M\to0-\\ N\to-2+0}}\int_N^M \frac{1}{\sqrt{1-(x+1)^2}}dx = \lim_{\substack{M\to0-\\ N\to-2+0}}\left[\sin^{-1}(x+1)\right]_N^M = \lim_{\substack{M\to0-\\ N\to-2+0}}\left\{\sin^{-1}(M+1)-\sin^{-1}(N+1)\right\} = \pi$$

(9) $\displaystyle\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}}dx = \int_{-1}^{0} \frac{1}{x^{\frac{2}{3}}}dx + \int_{0}^{1} \frac{1}{x^{\frac{2}{3}}}dx$

$$= \lim_{M\to 0-} \int_{-1}^{M} \frac{1}{x^{\frac{2}{3}}}dx + \lim_{N\to 0+} \int_{N}^{1} \frac{1}{x^{\frac{2}{3}}}dx$$

$$= \lim_{M\to 0-} \left[3x^{\frac{1}{3}}\right]_{-1}^{M} + \lim_{N\to 0+} \left[3x^{\frac{1}{3}}\right]_{N}^{1}$$

$$= \lim_{M\to 0-} \left(3M^{\frac{1}{3}} + 3\right) + \lim_{N\to 0+} \left(3 - 3N^{\frac{1}{3}}\right) = 6$$

(10) $\displaystyle\int_{0}^{2} \frac{1}{\sqrt{-x^2+2x+3}}dx = \int_{0}^{2} \frac{1}{\sqrt{4-(x-1)^2}}dx = \left[\sin^{-1}\frac{x-1}{2}\right]_{0}^{2} = \frac{\pi}{3}$

(11) $\displaystyle\int_{0}^{e} x\log x dx = \lim_{M\to 0+} \int_{M}^{e} x\log x$

$$= \lim_{M\to 0+} \left\{\left[\frac{x^2}{2}\log x\right]_{M}^{e} - \frac{1}{2}\int_{M}^{e} x dx\right\}$$

$$= \lim_{M\to 0+} \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_{M}^{e}$$

$$= \lim_{M\to 0+} \left(\frac{e^2}{2} - \frac{e^2}{4} - \frac{M^2}{2}\log M - \frac{M^2}{4}\right) = \frac{e^2}{2}$$

なお, $\displaystyle\lim_{M\to 0+} M^2 \log M = 0$ については, ロピタルの定理を用いて次のようにして求める.

$$\lim_{M\to 0+} M^2 \log M = \lim_{M\to 0+} \frac{\log M}{\frac{1}{M^2}} = \lim_{M\to 0+} \frac{\frac{1}{M}}{-\frac{2}{M^3}} = \lim_{M\to 0+} \left(-\frac{M^2}{2}\right) = 0$$

(12) $\displaystyle\int_{0}^{1} \frac{1+x^2}{\sqrt{1-x^2}}dx = \lim_{M\to 1-0} \int_{0}^{M} \frac{1+x^2}{\sqrt{1-x^2}}dx$

ここで, $x = \sin t$ とおくと, $dx = \cos t dt$,

x	$0 \to M$
t	$0 \to \sin^{-1} M$

.

よって, $N = \sin^{-1} M$ とすれば,

$$\begin{aligned}
\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}dx &= \lim_{M\to 1-0}\int_0^{\sin^{-1}M} \frac{1+\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t dt \\
&= \lim_{M\to 1-0}\int_0^{\sin^{-1}M} (1+\sin^2 t)dt \\
&= \lim_{M\to 1-0}\int_0^{\sin^{-1}M} \left(1+\frac{1-\cos 2t}{2}\right)dt \\
&= \lim_{N\to \frac{\pi}{2}-0}\int_0^{N} \left(\frac{3}{2}-\frac{1}{2}\cos 2t\right)dt \\
&= \lim_{N\to \frac{\pi}{2}-0}\left[\frac{3}{2}t-\frac{1}{4}\sin 2t\right]_0^N \\
&= \lim_{N\to \frac{\pi}{2}-0}\left(\frac{3}{2}N-\frac{1}{4}\sin 2N\right) = \frac{3}{4}\pi
\end{aligned}$$

(13) $$\begin{aligned}
\int_0^{\pi} \tan x dx &= \int_0^{\frac{\pi}{2}} \tan x dx + \int_{\frac{\pi}{2}}^{\pi} \tan x dx \\
&= \lim_{M\to \frac{\pi}{2}-0}\int_0^{M} \tan x dx + \lim_{N\to \frac{\pi}{2}+0}\int_N^{\pi} \tan x dx \\
&= \lim_{M\to \frac{\pi}{2}-0}[-\log|\cos x|]_0^M + \lim_{N\to \frac{\pi}{2}+0}[-\log|\cos x|]_N^{\pi} \\
&= \lim_{M\to \frac{\pi}{2}-0}(-\log|\cos M|) + \lim_{N\to \frac{\pi}{2}+0}\log|\cos N| = +\infty - \infty
\end{aligned}$$

よって発散. ゆえに広義積分は存在しない.

(14) $$\begin{aligned}
\int_{-1}^1 \frac{\log|x|}{\sqrt[3]{x}}dx &= \int_{-1}^0 \frac{\log|x|}{\sqrt[3]{x}}dx + \int_0^1 \frac{\log|x|}{\sqrt[3]{x}}dx \\
&= \lim_{M\to 0-}\int_{-1}^{M} \frac{\log(-x)}{\sqrt[3]{x}}dx + \lim_{N\to 0+}\int_N^1 \frac{\log|x|}{\sqrt[3]{x}}dx \\
&= \lim_{M\to 0-}\left\{\left[\frac{3}{2}x^{\frac{2}{3}}\log(-x)\right]_{-1}^M - \frac{3}{2}\int_{-1}^M x^{-\frac{1}{3}}dx\right\} \\
&\quad + \lim_{N\to 0+}\left\{\left[\frac{3}{2}x^{\frac{2}{3}}\log x\right]_N^1 - \frac{3}{2}\int_N^1 x^{-\frac{1}{3}}dx\right\} \\
&= \lim_{M\to 0-}\left[\frac{3}{2}x^{\frac{2}{3}}\log(-x) - \frac{9}{4}x^{\frac{2}{3}}\right]_{-1}^M + \lim_{N\to 0+}\left[\frac{3}{2}x^{\frac{2}{3}}\log x - \frac{9}{4}x^{\frac{2}{3}}\right]_N^1 \\
&= \lim_{M\to 0-}\left(\frac{3}{2}M^{\frac{2}{3}}\log(-M) - \frac{9}{4}M^{\frac{2}{3}} + \frac{9}{4}\right) \\
&\quad + \lim_{N\to 0+}\left(-\frac{9}{4} - \frac{3}{2}M^{\frac{2}{3}}\log M + \frac{9}{4}M^{\frac{2}{3}}\right) = 0
\end{aligned}$$

なお, $\lim_{x\to+0} x^{\frac{2}{3}}\log x = 0$ はロピタルの定理を用いて次のように求めた.

$$\lim_{x\to+0} x^{\frac{2}{3}}\log x = \lim_{x\to+0}\frac{\log x}{\frac{1}{x^{\frac{2}{3}}}}$$
$$= \lim_{x\to+0}\frac{\frac{1}{x}}{-\frac{2}{3x^{\frac{5}{3}}}} = \lim_{x\to+0}\left(-\frac{3}{2}x^{\frac{2}{3}}\right) = 0$$

(15) $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}dx = \lim_{M\to0+}\int_M^4 \frac{1}{\sqrt{x}}dx = \lim_{M\to0+}\left[2\sqrt{x}\right]_M^4 = \lim_{M\to0+}\left(4-2\sqrt{M}\right) = 4$

(16) $\displaystyle\int_1^\infty \frac{1}{x(1+x^2)}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x(1+x^2)}dx$

ここで,

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

とおく. 両辺を $x(1+x^2)$ 倍すると,

$$1 = A(1+x^2) + (Bx+C)x$$

この式の両辺に $x=0$ を代入すると $A=1$, $x=1$ を代入すると $1=2A+B+C$ より $B+C=-1$, $x=-1$ を代入すると $1=2A+B-C$ より $B-C=-1$. これより, $B=-1$, $C=0$. よって,

$$\int_1^\infty \frac{1}{x(1+x^2)}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x(1+x^2)}dx$$
$$= \lim_{M\to\infty}\int_1^M \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$
$$= \lim_{M\to\infty}\left[\log|x| - \frac{1}{2}\log(1+x^2)\right]_1^M$$
$$= \lim_{M\to\infty}\left(\log\frac{M}{\sqrt{1+M^2}} - \log\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log 2$$

(17) $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{x^2}dx = \lim_{M\to\infty}\int_1^M \frac{\tan^{-1}x}{x^2}dx$

$$= \lim_{M\to\infty}\left\{\left[-\frac{1}{x}\tan^{-1}x\right]_1^M + \int_1^M \frac{1}{x(1+x^2)}dx\right\}$$
$$= \lim_{M\to\infty}\left[-\frac{1}{x}\tan^{-1}x + \log|x| - \frac{1}{2}\log(1+x^2)\right]_1^M$$
$$= \lim_{M\to\infty}\left(-\frac{1}{M}\tan^{-1}M + \log\frac{M}{\sqrt{1+M^2}} + \tan^{-1}1 - \log\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\pi}{4} + \frac{1}{2}\log 2$$

ここで, $\displaystyle\int \frac{1}{x(1+x^2)}dx = \log|x| - \frac{1}{2}\log(1+x^2) + C$ は (16) の結果を利用した.

(18) $$\int_0^\infty \tan^{-1}\frac{1}{x}dx = \lim_{\substack{M\to\infty \\ N\to 0+}} \int_N^M \tan^{-1}\frac{1}{x}dx$$
$$= \lim_{\substack{M\to\infty \\ N\to 0+}} \left\{ \left[x\tan^{-1}\frac{1}{x}\right]_N^M - \int_N^M x\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)dx \right\}$$
$$= \lim_{\substack{M\to\infty \\ N\to 0+}} \left\{ \left[x\tan^{-1}\frac{1}{x}\right]_N^M + \int_N^M \frac{x}{x^2+1}dx \right\}$$
$$= \lim_{\substack{M\to\infty \\ N\to 0+}} \left[x\tan^{-1}\frac{1}{x} + \frac{1}{2}\log(x^2+1)\right]_N^M$$
$$= \lim_{\substack{M\to\infty \\ N\to 0+}} \left\{ \left(M\tan^{-1}\frac{1}{M} + \frac{1}{2}\log(M^2+1)\right) - \left(N\tan^{-1}\frac{1}{N} + \frac{1}{2}\log(N^2+1)\right) \right\} = +\infty$$

よって, 広義積分は存在しない.

2.20. $s \neq 1$ のとき,

$$\int_1^\infty \frac{1}{x^s}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x^s}dx = \lim_{M\to\infty}\left[\frac{1}{1-s}x^{1-s}\right]_1^M = \lim_{M\to\infty}\left(\frac{1}{1-s}M^{1-s} - \frac{1}{1-s}\right)$$

よって, 次の 2 通りに場合分けできる.

(i) $s > 1$ のとき. このとき $1-s<0$ より, $\displaystyle\int_1^\infty \frac{1}{x^s}dx$ は存在して,

$$\int_1^\infty \frac{1}{x^s}dx = \frac{1}{s-1}$$

(ii) $s < 1$ のとき. このとき $1-s>0$ より, $\displaystyle\int_1^\infty \frac{1}{x^s}dx$ は存在しない.

なお, $s=1$ のときは,

$$\int_1^\infty \frac{1}{x}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x}dx = \lim_{M\to\infty}[\log|x|]_1^M = \lim_{M\to\infty}\log M = \infty$$

となり, $\displaystyle\int_1^\infty \frac{1}{x^s}dx$ は存在しない.

一方, $\displaystyle\int_0^1 \frac{1}{x^s}dx = \lim_{M\to 0+}\int_M^1 \frac{1}{x^s}dx = \lim_{M\to 0+}\left[\frac{1}{1-s}x^{1-s}\right]_M^1 = \lim_{M\to 0+}\left(\frac{1}{1-s} - \frac{1}{1-s}M^{1-s}\right)$

よって, 次の 2 通りに場合分けできる.

(i) $s<1$ のとき. このとき $1-s>0$ より, $\displaystyle\int_0^1 \frac{1}{x^s}dx$ は存在して,

$$\int_0^1 \frac{1}{x^s}dx = \frac{1}{1-s}$$

(ii) $s>1$ のとき. このとき $1-s<0$ より, $\displaystyle\int_0^1 \frac{1}{x^s}dx$ は存在しない.

なお, $s=1$ のときは,

$$\int_0^1 \frac{1}{x}dx = \lim_{M\to 0+}\int_M^1 \frac{1}{x}dx = \lim_{M\to 0+}[\log|x|]_M^1 = \lim_{M\to 0+}(-\log M) = +\infty$$

となり, $\displaystyle\int_1^\infty \frac{1}{x^s}dx$ は存在しない.

2.21.

(1) $y=x^3-x^2+2$ とすれば, $y'=3x^2-2x$. これより, 点 $(1,2)$ における接線の方程式は

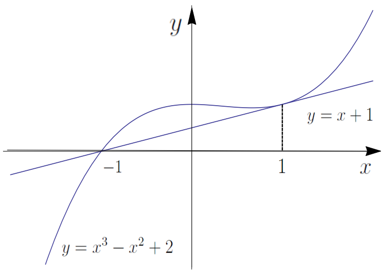
$$y-2=(x-1) \iff y=x+1$$

この直線と $y=x^3-x^2+2$ の交点の x 座標は,

$$\begin{aligned} x^3-x^2+2=x+1 &\iff x^3-x^2-x+1=0 \\ &\iff (x-1)^2(x+1)=0 \end{aligned}$$

より, $x=1,-1$.

また, 関数 $y=x^3-x^2+2$ と直線 $y=x+1$ のグラフは次のようになる.



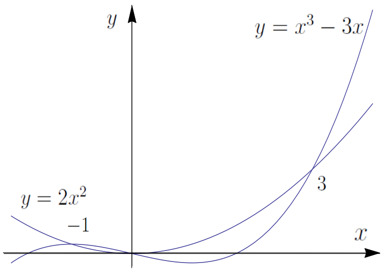
よって, 求める面積は,

$$\begin{aligned} \int_{-1}^1 \{(x^3-x^2+2)-(x+1)\}dx &= \int_{-1}^1 (x^3-x^2-x+1)dx \\ &= \left[\frac{1}{4}x^4-\frac{1}{3}x^3-\frac{1}{2}x^2+x\right]_{-1}^1 = \frac{4}{3} \end{aligned}$$

(2) 関数 $y=x^3-3x$ と $y=2x^2$ の交点の x 座標は

$$\begin{aligned} x^3-3x=2x^2 &\iff x^3-2x^2-3x=0 \\ &\iff x(x-3)(x+1)=0 \end{aligned}$$

より, $x=-1,0,3$. また, これらの関数のグラフは次のとおりである.



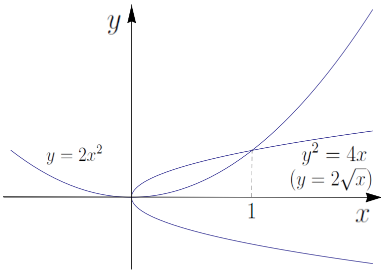
よって, 求める面積は

$$\int_{-1}^{0}(x^3-3x-2x^2)dx+\int_{0}^{3}(2x^2-x^3+3x)dx$$
$$=\left[\frac{1}{4}x^4-\frac{3}{2}x^2-\frac{2}{3}x^3\right]_{-1}^{0}+\left[\frac{2}{3}x^3-\frac{1}{4}x^4+\frac{3}{2}x^2\right]_{0}^{3}=\frac{71}{6}$$

(3) 関数 $y=2x^2$ と $y^2=4x$ の交点の x 座標は

$$4x^4=4x \iff x(x^3-1)=0$$

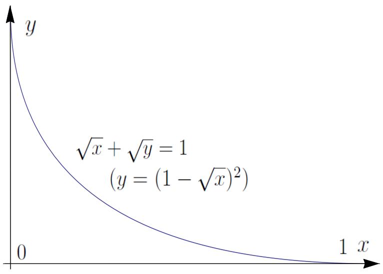
より, $x=0,1$. また, これらの関数のグラフは次のとおりである.



よって, 求める面積は

$$\int_{0}^{1}(2\sqrt{x}-2x^2)dx=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{3}x^3\right]_{0}^{1}=\frac{2}{3}$$

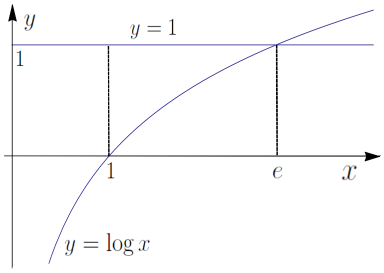
(4) $\sqrt{x}+\sqrt{y}=1$ のグラフは次の通りである.



よって, 図より求める面積は

$$\int_{0}^{1}(1-\sqrt{x})^2dx=\int_{0}^{1}(1-2\sqrt{x}+x)dx$$
$$=\left[x-\frac{4}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_{0}^{1}=\frac{1}{6}$$

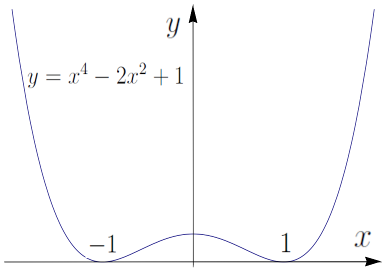
(5) $y = \log x$ と $y = 1$ のグラフは次の通りである.



よって, 図より求める面積は

$$\begin{aligned} 1 + \int_1^e (1 - \log x)dx &= 1 + [x]_1^e - \int_1^e \log x dx \\ &= 1 + [x]_1^e - \left\{ [x \log x]_1^e - \int_1^e dx \right\} \\ &= 1 + 2[x]_1^e - [x \log x]_1^e = e - 1 \end{aligned}$$

(6) $y = x^4 - 2x^2 + 1$ のグラフは次の通りである.



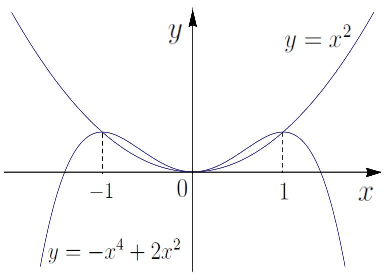
よって, 図より求める面積は

$$\int_{-1}^1 (x^4 - 2x^2 + 1)dx = \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x \right]_{-1}^1 = \frac{16}{15}$$

(7) $y = -x^4 + 2x^2$ と $y = x^2$ の交点の x 座標は

$$\begin{aligned} -x^4 + 2x^2 = x^2 &\iff x^4 - x^2 = 0 \\ &\iff x^2(x^2 - 1) = 0 \\ &\iff x = 0, \pm 1 \end{aligned}$$

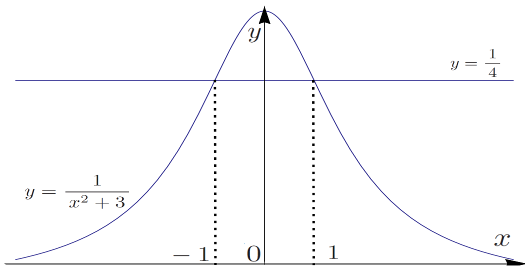
また, これらの関数のグラフは次の通りである.



よって, 図より求める面積は

$$2\int_0^1 (-x^4+2x^2-x^2)dx = 2\left[-\frac{1}{5}x^5+\frac{1}{3}x^3\right]_0^1 = \frac{4}{15}$$

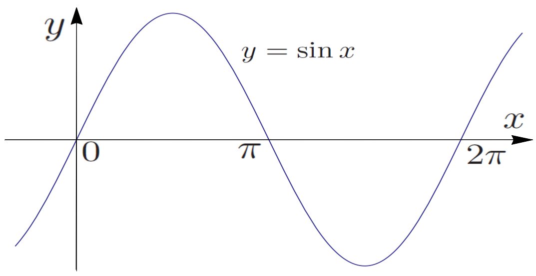
(8) $y=\dfrac{1}{x^2+3}$ と $y=\dfrac{1}{4}$ のグラフは次の通りである.



よって, 図より求める面積は

$$2\int_0^1 \left(\frac{1}{x^2+3}-\frac{1}{4}\right)dx = 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-\frac{1}{4}x\right]_0^1 = \frac{\pi}{3\sqrt{3}}-\frac{1}{2}$$

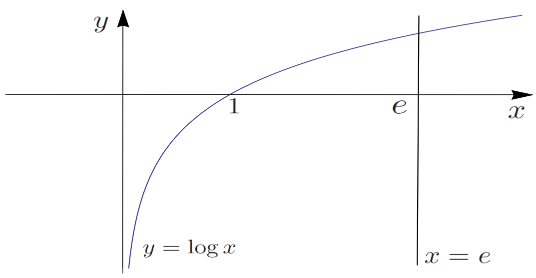
(9) $y=\sin x$ のグラフは次の通りである.



よって, 図より求める面積は

$$2\int_0^{\pi} \sin x dx = 2[-\cos x]_0^{\pi} = 4$$

(10) $y=\log x$ と $x=e$ のグラフは次の通りである.



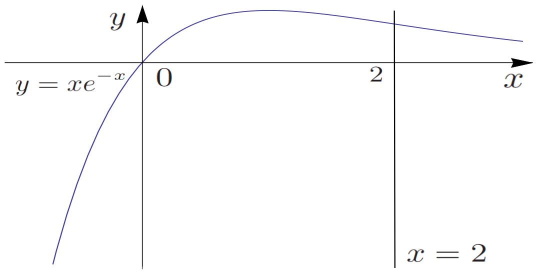
よって, 図より求める面積は

$$\int_1^e \log x dx = [x\log x]_1^e - \int_1^e dx = e-[x]_1^e = 1$$

(11) $y = xe^{-x}$ について, $y' = (1-x)e^{-x}$ より, その増減表は次の通りである.

x		1	
y'	$+$	0	$-$
y	$\nearrow$	$\dfrac{1}{e}$	$\searrow$

また, ロピタルの定理 (定理 1.30) より $\lim_{x\to-\infty} xe^{-x} = -\infty$, $\lim_{x\to\infty} xe^{-x} = 0$. よって, $y = xe^{-x}$ と $x = 2$ のグラフは次の通りである.



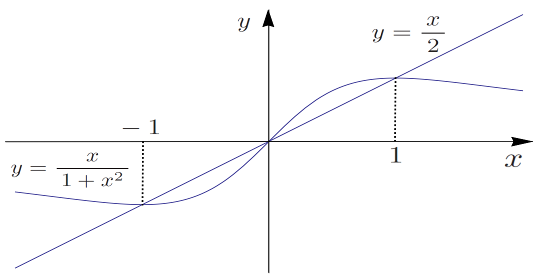
ゆえに, 求める面積は

$$\int_0^2 xe^{-x}dx = [-xe^{-x}]_0^2 - \int_0^2 (-e^{-x})dx$$
$$= [-xe^{-x} - e^{-x}]_0^2 = 1 - \frac{3}{e^2}$$

(12) $y = \dfrac{x}{1+x^2}$ について, $y' = \dfrac{1-x^2}{(1+x^2)^2}$ より, その増減表は次の通りである.

x		-1		1	
y'	$-$	0	$+$	0	$-$
y	$\searrow$	$-\dfrac{1}{2}$	$\nearrow$	$\dfrac{1}{2}$	$\searrow$

また, $\lim_{x\to\pm\infty} \dfrac{x}{1+x^2} = 0$ より, $y = \dfrac{x}{1+x^2}$ と $y = \dfrac{x}{2}$ のグラフは次の通りである.



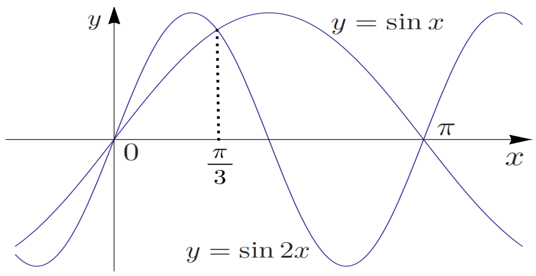
よって, 図より求める面積は

$$2\int_0^1 \left(\frac{x}{1+x^2} - \frac{x}{2}\right) dx = 2\left[\frac{1}{2}\log(1+x^2) - \frac{1}{4}x^2\right]_0^1 = \log 2 - \frac{1}{2}$$

(13) $y=\sin 2x$ と $y=\sin x$ の $0 \leqq x \leqq \pi$ における交点の x 座標は

$$\begin{aligned}\sin 2x=\sin x &\Longleftrightarrow 2\sin x\cos x=\sin x\\ &\Longleftrightarrow \sin x(2\cos x-1)=0\\ &\Longleftrightarrow \sin x=0, \cos x=\frac{1}{2}\\ &\Longleftrightarrow x=0, \pi, \frac{\pi}{3}\end{aligned}$$

また, $y=\sin 2x$ と $y=\sin x$ のグラフは次の通りである.

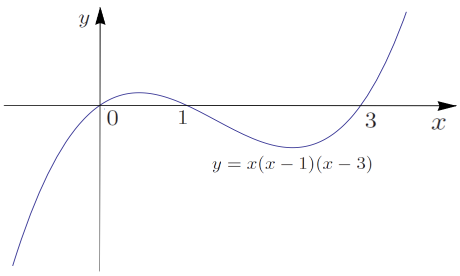


よって, 図より求める面積は

$$\int_0^{\frac{\pi}{3}}(\sin 2x-\sin x)dx+\int_{\frac{\pi}{3}}^{\pi}(\sin x-\sin 2x)dx$$

$$=\left[-\frac{1}{2}\cos 2x+\cos x\right]_0^{\frac{\pi}{3}}+\left[-\cos x+\frac{1}{2}\cos 2x\right]_{\frac{\pi}{3}}^{\pi}=\frac{5}{2}$$

(14) $y=x(x-1)(x-3)$ のグラフは次の通りである.



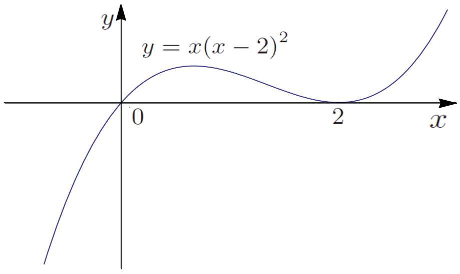
よって, 図より求める面積は

$$\int_0^1 x(x-1)(x-3)dx-\int_1^3 x(x-1)(x-3)dx$$

$$=\int_0^1(x^3-4x^2+3x)dx-\int_1^3(x^3-4x^2+3x)dx$$

$$=\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_0^1-\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_1^3=\frac{37}{12}$$

(15) $y = x(x-2)^2$ のグラフは次の通りである.



よって, 図より求める面積は

$$\int_0^2 x(x-2)^2 dx = \int_0^2 (x^3 - 4x^2 + 4x)dx$$
$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = \frac{4}{3}$$

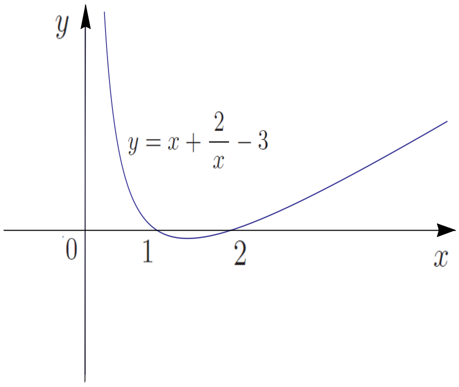
(16) $y = x + \dfrac{2}{x} - 3$ について, $y' = \dfrac{x^2-2}{x^2}$, $y'' = \dfrac{4}{x^3}$ である. よって, 関数 $y = x + \dfrac{2}{x} - 3$ の増減表は次の通りである.

x		$-\sqrt{2}$		0		$\sqrt{2}$	
y'	$+$	0	$-$		$-$	0	$+$
y''	$-$	$-$	$-$		$+$	$+$	$+$
y	↗	$-3-2\sqrt{2}$	↘		↘	$-3+2\sqrt{2}$	↗

関数 $y = x + \dfrac{2}{x} - 3$ の, x 軸との交点の x 座標は

$$x + \frac{2}{x} - 3 = 0 \iff x^2 - 3x + 2 = 0$$
$$\iff (x-1)(x-2) = 0 \iff x = 1, 2$$

また, $\displaystyle\lim_{x\to 0\pm}\left(x + \frac{2}{x} - 3\right) = \pm\infty$, $\displaystyle\lim_{x\to\pm\infty}\left(x + \frac{2}{x} - 3\right) = \pm\infty$. よって, 関数 $y = x + \dfrac{2}{x} - 3$ のグラフは次の通りである.



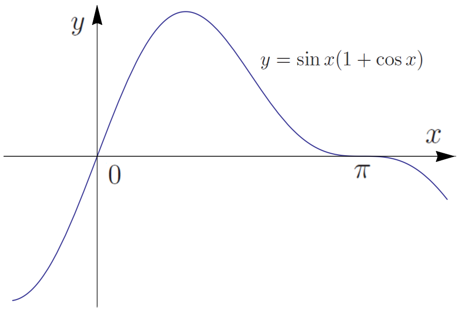
ゆえに, 図より求める面積は

$$-\int_1^2 \left(x+\frac{2}{x}-3\right)dx = -\left[\frac{1}{2}x^2+2\log|x|-3x\right]_1^2 = \frac{3}{2}-2\log 2$$

(17) $y = \sin x(1+\cos x)$ について, $y' = (\cos x+1)(2\cos x-1)$. よって, 関数 $y=\sin x(1+\cos x)$ の増減表は次の通りである.

x	0		$\frac{\pi}{3}$		π
y'	$+$	$+$	0	$-$	0
y	0	$\nearrow$	$\frac{3}{4}\sqrt{3}$	$\searrow$	0

よって, 関数 $y=\sin x(1+\cos x)$ のグラフは次の通りである.

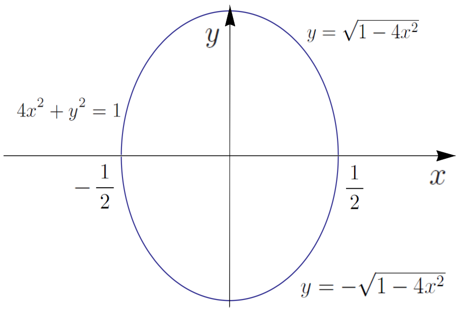


ゆえに, 図より求める面積は

$$\int_0^\pi \sin x(1+\cos x)dx = \int_0^\pi \left(\sin x+\frac{1}{2}\sin 2x\right)dx$$

$$= \left[-\cos x-\frac{1}{4}\cos 2x\right]_0^\pi = 2$$

(18) $4x^2+y^2=1$ のグラフは次の通りである.



よって, 図より求める面積は $2\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1-4x^2}dx$ である. ここで, $x=\frac{1}{2}\sin t$ とおくと, $dx=\frac{1}{2}\cos t dt$,

x	$-\frac{1}{2}\to\frac{1}{2}$
t	$-\frac{\pi}{2}\to\frac{\pi}{2}$

. よって,

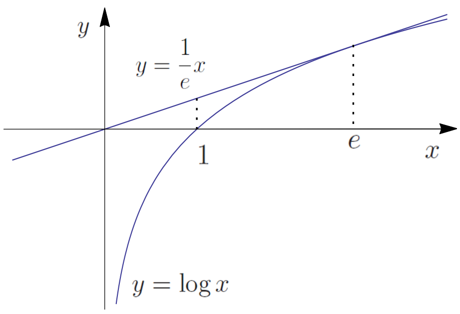
$$
\begin{aligned}
2\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1-4x^2}dx &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\cdot\frac{1}{2}\cos t dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt \\
&= \frac{1}{2}\left[t+\frac{1}{2}\sin 2t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}
\end{aligned}
$$

2.22.

(1) $y=\log x$ について, $y'=\frac{1}{x}$. これより, $y=\log x$ の接線の方程式は, 接点の座標を $(c,\log c)$ とすれば,

$$y-\log c=\frac{1}{c}(x-c) \iff y=\frac{1}{c}x-1+\log c$$

これが原点を通るので, $c=e$, すなわち, 接線の方程式は $y=\frac{1}{e}x$ となる. この接線と $y=\log x$ のグラフは次の通りである.



よって, 図より求める面積は,

$$
\begin{aligned}
&1 \times \frac{1}{e} \times \frac{1}{2} + \int_1^e \left(\frac{1}{e}x - \log x \right) dx \\
&= \frac{1}{2e} + \left[\frac{1}{2e}x^2 \right]_1^e - \left\{ [x \log x]_1^e - \int_1^e dx \right\} \\
&= \frac{1}{2e} + \left(\frac{e}{2} - \frac{1}{2e} \right) - [x \log x - x]_1^e = \frac{e}{2} - 1
\end{aligned}
$$

(2) $y = \dfrac{\log x}{x}$ とすると, $y' = \dfrac{1 - \log x}{x^2}$. これより, 原点から $y = \dfrac{\log x}{x}$ へ引いた接線の, 接点の x 座標を c とすると求める接線の方程式は

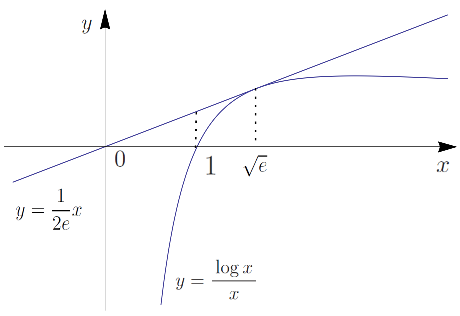
$$
y - \frac{\log c}{c} = \frac{1 - \log c}{c^2}(x - c) \iff y = \frac{1 - \log c}{c^2}x - \frac{1 - 2\log c}{c}.
$$

この接線が原点を通るので, $c = \sqrt{e}$, すなわち, 接線の方程式は $y = \dfrac{1}{2e}x$.

いっぽう, $y = \dfrac{\log x}{x}$ の増減表は次の通りである.

x	0		e	
y'	＼	$+$	0	$-$
y	＼	$\nearrow$	$\frac{1}{e}$	$\searrow$

よって, この関数と接線のグラフは次の通りである.



ゆえに, 求める面積は

$$
\begin{aligned}
1 \times \frac{1}{2e} \times \frac{1}{2} + \int_1^{\sqrt{e}} \left(\frac{x}{2e} - \frac{\log x}{x} \right) dx &= \frac{1}{4e} + \left[\frac{x^2}{4e} \right]_1^{\sqrt{e}} - \int_1^{\sqrt{e}} \frac{\log x}{x} dx \\
&= \frac{1}{4} - \int_1^{\sqrt{e}} \frac{\log x}{x} dx
\end{aligned}
$$

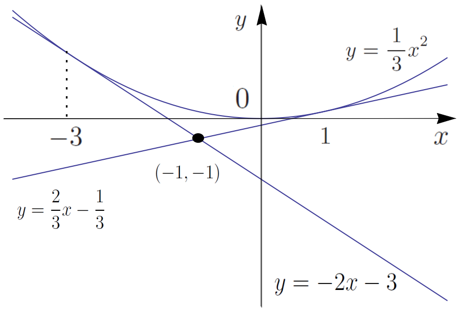
ここで, $\log x = t$ とおくと, $\dfrac{1}{x}dx = dt$,

x	$1 \to \sqrt{e}$
t	$0 \to \dfrac{1}{2}$

. よって,

$$(\text{与式}) = \frac{1}{4} - \int_0^{\frac{1}{2}} tdt = \frac{1}{4} - \left[\frac{1}{2}t^2\right]_0^{\frac{1}{2}} = \frac{1}{8}$$

(3) $y = \dfrac{1}{3}x^2$ より $y' = \dfrac{2}{3}x$. よって, 点 A における接線の方程式は $y = \dfrac{2}{3}x - \dfrac{1}{3}$, 点 B における接線の方程式は $y = -2x - 3$. また, これらの接線の交点の座標は $(-1, -1)$ である. そして, これらの関数のグラフは次の通りである.



よって, 図より求める面積は

$$\int_{-3}^{-1} \left(\frac{1}{3}x^2 + 2x + 3\right) dx + \int_{-1}^{1} \left(\frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3}\right) dx$$

$$= \left[\frac{1}{9}x^3 + x^2 + 3x\right]_{-3}^{-1} + \left[\frac{1}{9}x^3 - \frac{1}{3}x^2 + \frac{1}{3}x\right]_{-1}^{1} = \frac{16}{9}$$

2.23.

(1) $0 \leqq t \leqq 2$ において, $y = 2t - t^2 \geqq 0$, $x' = 2 > 0$ より, 求める面積は,

$$\int_0^2 (2t - t^2) \cdot 2dt = 2\left[t^2 - \frac{1}{3}t^3\right]_0^2 = \frac{8}{3}$$

(2) $-1 \leqq t \leqq 3$ において $y = t^2 - 2t - 3 \leqq 0$, $x' = -1 < 0$ より, 求める面積は

$$-\int_{-1}^{3} (t^2 - 2t - 3) \cdot |-1| dt = -\left[\frac{1}{3}t^3 - t^2 - 3t\right]_{-1}^{3} = \frac{32}{3}$$

(3) $0 \leqq \theta \leqq 2\pi$ において, $y = 1 - \cos\theta \geqq 0$, $x' = 1 - \cos\theta \geqq 0$ より, 求める面積は,

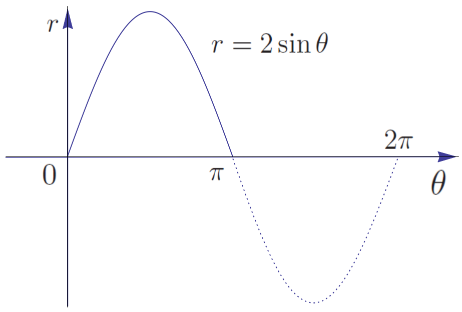
$$\begin{aligned}\int_0^{2\pi} (1-\cos\theta)^2 d\theta &= \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi\end{aligned}$$

(4) $0 \leqq \theta \leqq \dfrac{\pi}{2}$ における面積を求めて 4 倍すればよい. $0 \leqq \theta \leqq \dfrac{\pi}{2}$ において, $y = \cos^3\theta \geqq 0$, $x' = 3\sin^2\theta\cos\theta \geqq 0$. よって, 求める面積は, 例題 2.16 の結果を使って,

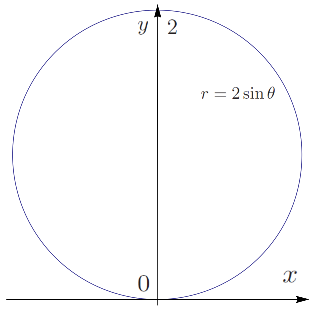
$$\begin{aligned}4\int_0^{\frac{\pi}{2}} \cos^3\theta \cdot 3\sin^2\theta\cos\theta d\theta &= 12\int_0^{\frac{\pi}{2}} \cos^4\theta(1-\cos^2\theta)d\theta \\ &= 12\int_0^{\frac{\pi}{2}} (\cos^4\theta - \cos^6\theta)d\theta \\ &= 12\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{3}{8}\pi\end{aligned}$$

2.24.

(1) 最初に $r = 2\sin\theta$ のグラフを (θ, r)-平面に描く. $r > 0$ に注意すると次のようになる.



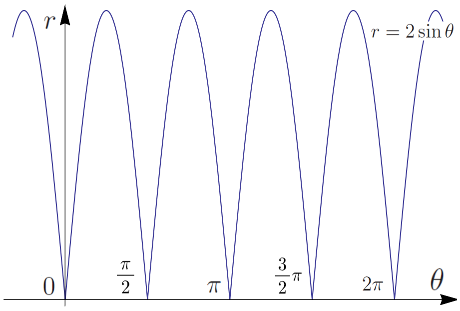
これを, θ は x 軸とのなす角, r は原点からの距離であることに注意して, (x, y)-平面にグラフを描くと次のようにになる.



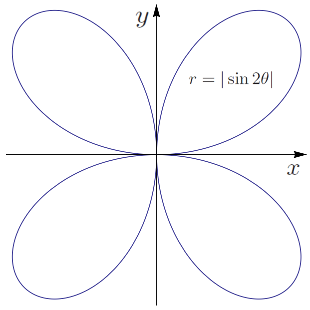
よって, この曲線で囲まれる部分の面積は,

$$\frac{1}{2}\int_0^{\pi}(2\sin\theta)^2 d\theta = 2\int_0^{\pi}\sin^2\theta d\theta$$
$$= \int_0^{\pi}(1-\cos 2\theta)d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi$$

(2) 最初に $r = |\sin 2\theta|$ のグラフを (θ, r)-平面に描くと次のようになる.



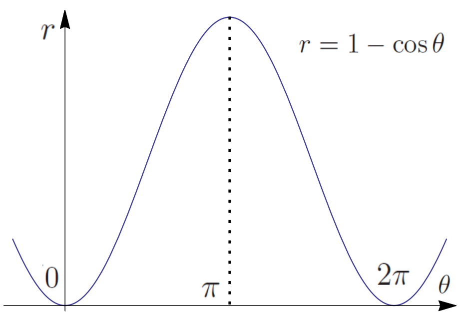
これを, θ は x 軸とのなす角, r は原点からの距離であることに注意して, (x, y)-平面にグラフを描くと次のようにになる.



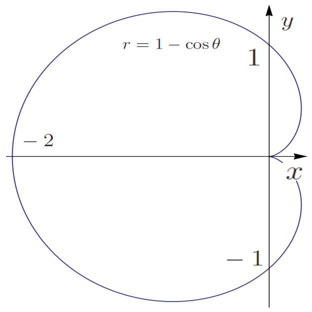
よって, この曲線で囲まれる部分の面積は,

$$4 \times \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\theta d\theta = \int_0^{\frac{\pi}{2}}(1-\cos 4\theta)d\theta$$
$$= \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

(3) 最初に $r = 1 - \cos\theta$ のグラフを (θ, r)-平面に描くと次のようになる.



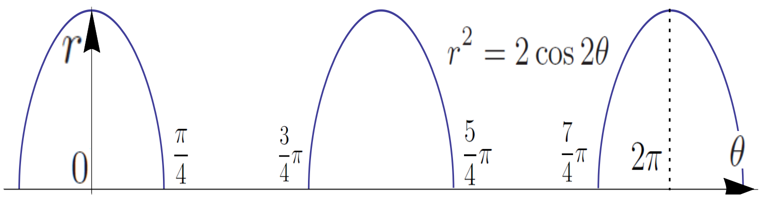
これを, θ は x 軸とのなす角, r は原点からの距離であることに注意して, (x,y)-平面にグラフを描くと次のようになる.



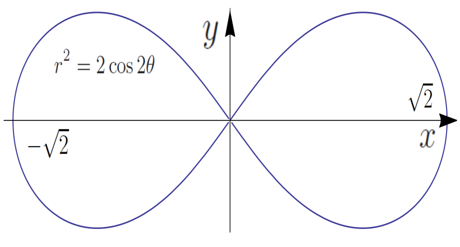
よって, この曲線で囲まれる部分の面積は,

$$
\begin{aligned}
2\times\frac{1}{2}\int_0^{\pi}(1-\cos\theta)^2 d\theta &= \int_0^{\pi}(1-2\cos\theta+\cos^2\theta)d\theta \\
&= \int_0^{\pi}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\
&= \int_0^{\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta \\
&= \left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi}=\frac{3}{2}\pi
\end{aligned}
$$

(4) 最初に $r^2=2\cos 2\theta$ のグラフを (θ,r)-平面に描く. $r^2\geqq 0$ であることに注意すると次のようになる.



これを, θ は x 軸とのなす角, r は原点からの距離であることに注意して, (x,y)-平面にグラフを描くと次のようになる.



よって, この曲線で囲まれる部分の面積は,

$$4\times\frac{1}{2}\int_0^{\frac{\pi}{4}}2\cos 2\theta d\theta = 4\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}=2$$

2.25.

(1) $y' = x$ より, 求める曲線の長さは $\displaystyle\int_0^1 \sqrt{1+x^2}dx$ である. ここで, $\sqrt{1+x^2} = t - x$ とおくと,

$$\sqrt{1+x^2} = t - x \Longrightarrow 1 + x^2 = t^2 - 2tx + x^2$$
$$\Longleftrightarrow x = \frac{t^2-1}{2t} = \frac{t}{2} - \frac{1}{2t}$$

これより

$$dx = \left(\frac{1}{2} + \frac{1}{2t^2}\right)dt = \frac{t^2+1}{2t^2}dt$$

x	$0 \to 1$
t	$1 \to 1+\sqrt{2}$

. また,

$$\sqrt{1+x^2} = t - x = t - \frac{t^2-1}{2t} = \frac{t^2+1}{2t}$$

よって,

$$\begin{aligned}
\int_0^1 \sqrt{1+x^2}dx &= \int_1^{1+\sqrt{2}} \frac{t^2+1}{2t} \cdot \frac{t^2+1}{2t^2}dt \\
&= \frac{1}{4}\int_1^{1+\sqrt{2}} \left(t + \frac{2}{t} + \frac{1}{t^3}\right)dt \\
&= \frac{1}{4}\left[\frac{1}{2}t^2 + 2\log|t| - \frac{1}{2t^2}\right]_1^{1+\sqrt{2}} \\
&= \frac{1}{8}\left[(1+\sqrt{2})^2 + 4\log(1+\sqrt{2}) - \frac{1}{(1+\sqrt{2})^2}\right] \\
&= \frac{1}{8}\left[(1+\sqrt{2})^2 + 4\log(1+\sqrt{2}) - (1-\sqrt{2})^2\right] = \frac{1}{2}\left\{\sqrt{2} + \log(1+\sqrt{2})\right\}
\end{aligned}$$

(2) $y' = -\dfrac{\sin x}{\cos x} = -\tan x$ より, 求める曲線の長さは

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}dx &= \int_0^{\frac{\pi}{3}} \frac{1}{\cos x}dx \\
&= \int_0^{\frac{\pi}{3}} \frac{\cos x}{\cos^2 x}dx = \int_0^{\frac{\pi}{3}} \frac{\cos x}{1-\sin^2 x}dx
\end{aligned}$$

ここで, $\sin x = t$ とおくと, $\cos x dx = dt$,

x	$0 \to \frac{\pi}{3}$
t	$1 \to \frac{\sqrt{3}}{2}$

. よって,

$$\begin{aligned}与式 &= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2} dt \\ &= -\frac{1}{2} \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt \\ &= -\frac{1}{2} \left[\log|t-1| - \log|t+1| \right]_0^{\frac{\sqrt{3}}{2}} = \log(2+\sqrt{3})\end{aligned}$$

(3) $x' = -3\cos^2\theta \sin\theta$, $y' = 3\sin^2\theta\cos\theta$ より, 求める曲線の長さは

$$\begin{aligned}&\int_0^{2\pi} \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2} d\theta \\ &= 3\int_0^{2\pi} \sqrt{\sin^2\theta\cos^2\theta} d\theta \\ &= 12\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta = 6\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = 3[-\cos 2\theta]_0^{\frac{\pi}{2}} = 6\end{aligned}$$

(4) $x' = e^t\cos 2\pi t - 2\pi e^t \sin 2\pi t$, $y' = e^t \sin 2\pi t + 2\pi e^t \cos 2\pi t$ より, 求める曲線の長さは

$$\begin{aligned}&\int_0^{\frac{3}{2}} \sqrt{(e^t\cos 2\pi t - 2\pi e^t\sin 2\pi t)^2 + (e^t\sin 2\pi t + 2\pi e^t \cos 2\pi t)^2} dt \\ &= \sqrt{1+4\pi^2} \int_0^{\frac{3}{2}} e^t dt = \sqrt{1+4\pi^2}[e^t]_0^{\frac{3}{2}} = \sqrt{1+4\pi^2}(e^{\frac{3}{2}} - 1)\end{aligned}$$

2.26.

(1) $r' = 2\theta$ より, 求める曲線の長さは

$$\int_0^{2\pi} \sqrt{\theta^4 + (2\theta)^2} d\theta = \int_0^{2\pi} \theta\sqrt{4+\theta^2} d\theta$$

ここで, $4+\theta^2 = t$ とおくと, $2\theta d\theta = dt$,

θ	$0 \to 2\pi$
t	$4 \to 4+4\pi^2$

. よって,

$$\begin{aligned}\int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta &= \frac{1}{2}\int_4^{4+4\pi^2} \sqrt{t} dt \\ &= \frac{1}{2}\left[\frac{2}{3} t^{\frac{3}{2}}\right]_4^{4+4\pi^2} = \frac{8}{3}\left\{(1+\pi^2)^{\frac{3}{2}} - 1\right\}\end{aligned}$$

(2) $r' = -\sin\theta$ より, 求める曲線の長さは

$$\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}d\theta = \int_0^{2\pi} \sqrt{2+2\cos\theta}d\theta$$
$$= \int_0^{2\pi} \sqrt{4\cos^2\frac{\theta}{2}}d\theta$$
$$= 2\int_0^{\pi} \cos\frac{\theta}{2}d\theta - 2\int_{\pi}^{2\pi} \cos\frac{\theta}{2}d\theta$$
$$= 2\left[2\sin\frac{\theta}{2}\right]_0^{\pi} - 2\left[2\sin\frac{\theta}{2}\right]_{\pi}^{2\pi} = 8$$

(3) $r' = \sin^2\dfrac{\theta}{3}\cos\dfrac{\theta}{3}$ より, 求める曲線の長さは

$$\int_0^{3\pi} \sqrt{\sin^6\frac{\theta}{3} + \left(\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\right)^2}d\theta = \int_0^{3\pi} \sqrt{\sin^4\frac{\theta}{3}}d\theta$$
$$= \int_0^{3\pi} \sin^2\frac{\theta}{3}d\theta$$
$$= \int_0^{3\pi} \frac{1-\cos\frac{2}{3}\theta}{2}d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{3}{2}\sin\frac{2}{3}\theta\right]_0^{3\pi} = \frac{3}{2}\pi$$

(4) $r' = e^{\theta}$ より, 求める曲線の長さは

$$\int_0^{2\pi} \sqrt{e^{2\theta}+e^{2\theta}}d\theta = \sqrt{2}\int_0^{2\pi} e^{\theta}d\theta = \sqrt{2}(e^{2\pi}-1)$$

(5) $r' = 4\cos^3\dfrac{\theta}{4}\left(-\sin\dfrac{\theta}{4}\right)\cdot\dfrac{1}{4} = -\cos^3\dfrac{\theta}{4}\sin\dfrac{\theta}{4}$ より, 求める曲線の長さは

$$\int_0^{2\pi} \sqrt{\cos^8\frac{\theta}{4} + \cos^6\frac{\theta}{4}\sin^2\frac{\theta}{4}}d\theta = \int_0^{2\pi} \sqrt{\cos^6\frac{\theta}{4}}d\theta$$
$$= \int_0^{2\pi} \cos^3\frac{\theta}{4}d\theta = \int_0^{2\pi} \cos\frac{\theta}{4}\left(1-\sin^2\frac{\theta}{4}\right)d\theta$$

ここで, $\sin\dfrac{\theta}{4} = t$ とおくと, $\dfrac{1}{4}\cos\dfrac{\theta}{4}d\theta = dt$,

θ	$0 \to 2\pi$
t	$0 \to 1$

. よって,

$$与式 = \int_0^1 4(1-t^2)dt = \left[4t - \frac{4}{3}t^3\right]_0^1 = \frac{8}{3}$$

(6) $r' = -e^{-\theta}$ より, 求める曲線の長さは

$$\begin{aligned}\int_0^{\infty} \sqrt{e^{-2\theta}+e^{-2\theta}}d\theta &= \lim_{M\to+\infty}\int_0^M \sqrt{2}e^{-\theta}d\theta \\ &= \lim_{M\to+\infty}\sqrt{2}\left[-e^{-\theta}\right]_0^M \\ &= \lim_{M\to+\infty}\sqrt{2}\left(1-e^{-M}\right) = \sqrt{2}\end{aligned}$$

2.27. 定理 2.22 $\Longrightarrow$ 定理 2.21, 定理 2.21 $\Longrightarrow$ 定理 2.22, 定理 2.22 $\Longrightarrow$ 定理 2.23, 定理 2.23 $\Longrightarrow$ 定理 2.21 をそれぞれ示す.

(1) 定理 2.22 $\Longrightarrow$ 定理 2.21 を示す.
定理 2.22 において, $x = x$, $y = f(x)$ とおけば, 定理 2.22 より $a \leqq x \leqq b$ における曲線の長さは

$$\int_a^b \sqrt{(x')^2+\{f'(x)\}^2}dx = \int_a^b \sqrt{1+f'(x)^2}dx$$

となり, 定理 2.21 が得られた.

(2) 定理 2.21 $\Longrightarrow$ 定理 2.22 を示す.
$x = x(t)$, $y = y(t)$ とすると, $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$. また, $a \leqq x \leqq b$ のとき $\alpha \leqq t \leqq \beta$ とすれば, 定理 2.21 と定理 2.15' より

$$\int_a^b \sqrt{1+f'(x)^2}dx = \int_\alpha^\beta \sqrt{1+\left(\frac{y'(t)}{x'(t)}\right)^2}\left|x'(t)\right|dt = \int_\alpha^\beta \sqrt{x'(t)^2+y'(t)^2}dt$$

となり, 定理 2.22 が得られた.

(3) 定理 2.22 $\Longrightarrow$ 定理 2.23 を示す.
$r = f(\theta)$ より, この曲線上の点 (x, y) を極座標表示すると

$$\begin{cases} x(\theta) &= r\cos\theta &= f(\theta)\cos\theta \\ y(\theta) &= r\sin\theta &= f(\theta)\sin\theta \end{cases}$$

これより,

$$\begin{cases} x'(\theta) &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ y(\theta) &= f'(\theta)\sin\theta + f(\theta)\cos\theta \end{cases}$$

よって, 定理 2.22 より $\alpha \leqq \theta \leqq \beta$ における曲線の長さは

$$\begin{aligned}&\int_\alpha^\beta \sqrt{\{f'(\theta)\cos\theta - f(\theta)\sin\theta\}^2+\{f'(\theta)\sin\theta + f(\theta)\cos\theta\}^2}d\theta \\ =&\int_\alpha^\beta \sqrt{f(\theta)^2+f'(\theta)^2}d\theta\end{aligned}$$

となり, 定理 2.23 を得る.

(4) 定理 2.23 $\Longrightarrow$ 定理 2.21 を示す.

$y=f(x)$ のとき, この曲線上の点 (x,y) を極座標表示する, すなわち, $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$.

このとき, $r\sin\theta=f(r\cos\theta)$ が成り立つ. ここで, r を θ の関数とみて両辺を θ で微分すると,

$$\begin{aligned} & r'\sin\theta+r\cos\theta=f'(r\cos\theta)\{r'\cos\theta-r\sin\theta\} \\ \Longleftrightarrow & r'\{\sin\theta-f'(r\cos\theta)\cos\theta\}=-r\{\cos\theta+f'(r\cos\theta)\sin\theta\} \\ \Longleftrightarrow & r'=-r\frac{\cos\theta+f'(r\cos\theta)\sin\theta}{\sin\theta-f'(r\cos\theta)\cos\theta} \end{aligned}$$

よって,

$$\begin{aligned} r^2+(r')^2 &= r^2+r^2\frac{\{\cos\theta+f'(r\cos\theta)\sin\theta\}^2}{\{\sin\theta-f'(r\cos\theta)\cos\theta\}^2} \\ &= r^2\frac{1+f'(r\cos\theta)^2}{\{\sin\theta-f'(r\cos\theta)\cos\theta\}^2} \end{aligned}$$

よって, $\alpha \leqq \theta \leqq \beta$ としたとき, この曲線の長さは, 定理 2.23 より

$$\int_\alpha^\beta \sqrt{r^2+(r')^2}d\theta=\int_\alpha^\beta r\sqrt{\frac{1+f'(r\cos\theta)^2}{\{\sin\theta-f'(r\cos\theta)\cos\theta\}^2}}d\theta$$

ここで, $x=r\cos\theta$ とおくと,

$$\begin{aligned} dx &= (r'\cos\theta-r\sin\theta)d\theta \\ &= \left\{-r\frac{\cos\theta+f'(r\cos\theta)\sin\theta}{\sin\theta-f'(r\cos\theta)\cos\theta}\cos\theta-r\sin\theta\right\}d\theta \\ &= -r\frac{1}{\sin\theta-f'(r\cos\theta)\cos\theta}d\theta \end{aligned}$$

また, $\alpha \leqq \theta \leqq \beta$ のとき $a \leqq x \leqq b$ とすれば, 定理 2.15' から

$$\begin{aligned} 与式 &= \int_a^b r\sqrt{\frac{1+f'(r\cos\theta)^2}{\{\sin\theta-f'(r\cos\theta)\cos\theta\}^2}}d\theta\cdot\left|\frac{\sin\theta-f'(r\cos\theta)\cos\theta}{-r}\right|dx \\ &= \int_a^b \sqrt{1+f'(x)}dx \end{aligned}$$

となり, 定理 2.21 を得た.

以上のことから, 定理 2.21, 定理 2.22, 定理 2.23 は互いに導きあえることが示された.

2.28.

(1) 回転体の体積は

$$\pi\int_0^1 x^6dx=\pi\left[\frac{1}{7}x^7\right]_0^1=\frac{1}{7}\pi$$

一方, $y' = 3x^2$ より, 回転体の表面積は

$$2\pi \int_0^1 x^3\sqrt{1+(3x^2)^2}dx = 2\pi \int_0^1 x^3\sqrt{1+9x^4}dx$$

ここで, $1+9x^4 = t$ とおくと, $36x^3dx = dt$,

x	$0 \to 1$
t	$1 \to 10$

. よって,

$$\begin{aligned} 2\pi \int_0^1 x^3\sqrt{1+9x^4}dx &= 2\pi \int_1^{10} \frac{1}{36}\sqrt{t}dt \\ &= \frac{\pi}{18}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^{10} = \frac{\pi}{27}(10\sqrt{10}-1) \end{aligned}$$

(2) 回転体の体積と表面積は, 0 から $\dfrac{\pi}{2}$ までの部分を求めて 2 倍すればよい.

回転体の体積は

$$2\pi \int_0^{\frac{\pi}{2}} \cos^2 x dx = \pi \int_0^{\frac{\pi}{2}} (1+\cos 2x)dx = \pi\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{2}$$

一方, $y' = -\sin x$ より, 回転体の表面積は

$$2 \times 2\pi \int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin^2 x}dx$$

ここで, $\sin x = t$ とおくと, $\cos x dx = dt$,

x	$0 \to \frac{\pi}{2}$
t	$0 \to 1$

. よって,

$$2 \times 2\pi \int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin^2 x}dx = 4\pi \int_0^1 \sqrt{1+t^2}dt$$

さらに, $\sqrt{1+t^2} = u - t$ とおくと,

$$\begin{aligned} \sqrt{1+t^2} = u - t &\Longrightarrow 1 + t^2 = u^2 - 2tu + t^2 \\ &\Longleftrightarrow t = \frac{u^2-1}{2u} = \frac{u}{2} - \frac{1}{2u} \end{aligned}$$

より, $dt = \left(\dfrac{1}{2} + \dfrac{1}{2u^2}\right)du = \dfrac{u^2+1}{2u^2}du$. また,

t	$0 \to 1$
u	$1 \to 1+\sqrt{2}$

,

$$\sqrt{1+t^2} = u - t = \frac{u^2+1}{2u}$$

よって,

$$
\begin{aligned}
\text{与式} &= 4\pi \int_0^1 \sqrt{1+t^2}dt \\
&= 4\pi \int_1^{1+\sqrt{2}} \frac{u^2+1}{2u} \cdot \frac{u^2+1}{2u^2} du \\
&= \pi \int_1^{1+\sqrt{2}} \left(u + \frac{2}{u} + \frac{1}{u^3} \right) du \\
&= \pi \left[\frac{1}{2}u^2 + 2\log u - \frac{1}{2u^2} \right]_1^{1+\sqrt{2}} \\
&= \frac{1}{2}\pi \left\{ (1+\sqrt{2})^2 + 4\log(1+\sqrt{2}) - \frac{1}{(1+\sqrt{2})^2} \right\} \\
&= \frac{1}{2}\pi \left\{ (1+\sqrt{2})^2 + 4\log(1+\sqrt{2}) - (1-\sqrt{2})^2 \right\} \\
&= 2\pi \left\{ \sqrt{2} + \log(1+\sqrt{2}) \right\}
\end{aligned}
$$