Numerical radius inequalities related to the geometric means of negative power

The Fifteenth Workshop on Numerical Ranges and Numerical Radii (WONRA) 2019
Toyo University, Kawagoe Campus

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June 21–24, 2019

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Motivation

Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, we write $A \geq 0$ if A is positive semidefinite and $A > 0$ if A is positive definite, that is, A is positive and invertible. For two Hermitian matrices A and B, we write $A \geq B$ if $A - B \geq 0$, and it is called the Löwner partial ordering. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if $|||UXV||| = |||X|||$ for all $X \in \mathbb{M}_n$ and unitary U, V.
For any matrix $A \in \mathbb{M}_n$, the numerical radius $w(A)$ is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : \|x\| = 1, x \in \mathbb{C}^n\}.$$

Then the numerical radius is unitarily similar: $w(U^*AU) = w(A)$ for all unitary U.

Motivation

Geometric means $A^{1-\alpha}B^{\alpha}$, $A \sharp_\alpha B$ and $A \diamondsuit_\alpha B$

Let A and B be positive definite matrices. The α-geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\alpha} A^{\frac{1}{2}} \qquad \text{for all } \alpha \in [0,1].$$

The chaotic geometric mean $A \diamondsuit_\alpha B$ is defined by

$$A \diamondsuit_\alpha B = e^{(1-\alpha)\log A + \alpha \log B} \qquad \text{for all } \alpha \in \mathbb{R}.$$

If A and B commute, then $A \sharp_\alpha B = A \diamondsuit_\alpha B = A^{1-\alpha}B^{\alpha}$ for all $\alpha \in [0,1]$. However, we have no relation among $A \sharp_\alpha B$, $A \diamondsuit_\alpha B$ and $A^{1-\alpha}B^{\alpha}$ for all $\alpha \in [0,1]$ under the Löwner partial order.

Remark:

$$A \diamondsuit_\alpha B = \lim_{p \to 0} ((1-\alpha)A^p + \alpha B^p)^{\frac{1}{p}}.$$

Motivation

Specht ratio

Specht estimated the upper bound of the arithmeticmean by the geometric one for positive numbers: For $x_1, \ldots, x_n \in [m, M]$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} \leq S(h) \sqrt[n]{x_1 x_2 \cdots x_n}$$

where $h = \frac{M}{m}$ and the Specht ratio is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \ (h \neq 1) \quad \text{and} \quad S(1) = 1$$

Motivation

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Comparison between $A \sharp_\alpha B$ and $A \diamondsuit_\alpha B$

Let A, B be positive definite matrices with $mI \le A, B \le MI$. Then

$$S(h)^{-1} A \diamondsuit_\alpha B \le A \sharp_\alpha B \le S(h) A \diamondsuit_\alpha B$$

Motivation

Norm inequality for geometric means

Let A, B be positive definite matrices. Then for every unitarily invariant norm $|||\cdot|||$

$$|||A \natural_\alpha B||| \le |||A \diamondsuit_\alpha B||| \le \left|\left|\left|A^{1-\alpha} B^\alpha\right|\right|\right| \quad \text{for all } \alpha \in [0,1].$$

Motivation

Norm inequality for geometric means

Let A, B be positive definite matrices. Then for every unitarily invariant norm $|||\cdot|||$

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We have the following numerical radius inequalities:

Numerical dadius inequality for geometric means

Let A, B be positive definite matrices. Then

$$w(A \sharp_\alpha B) \le w(A \diamondsuit_\alpha B) \le w(A^{1-\alpha}B^\alpha) \quad \text{for all } \alpha \in [0,1],$$

Motivation

The purpose of this talk

We discuss numerical radius inequalities related to the geometric means. Though the numerical radius is not unitarily invariant norm, the numerical radius is unitarily similar. In this talk, we show numerical radius inequalities related to the geometric means of negative power for positive definite matrices.

Contents

Geometric mean of negative power

Geometric mean of negative power

The β-quasi geometric mean $A \natural_\beta B$ is defined by

$$A \natural_\beta B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\beta A^{\frac{1}{2}} \qquad \text{for all } \beta \in [-1, 0),$$

whose formula is the same as $\sharp_\alpha$.
The chaotic geometric mean $A \diamondsuit_\beta B$ is defined by

$$A \diamondsuit_\beta B = e^{(1-\beta)\log A + \beta \log B} \qquad \text{for all } \beta \in [-1, 0).$$

The geometric mean is $A^{1-\beta} B^\beta$ for $\beta \in [-1, 0)$.

Geometric mean of negative power

Geometric mean of negative power

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$$\|A \sharp_\alpha B\| \le \|A \diamondsuit_\alpha B\| \le \left\| A^{1-\alpha} B^\alpha \right\| \qquad \text{for all } \alpha \in [0, 1].$$

Geometric mean of negative power

Ando-Hiai inequality

Let A, B be positive definite matrices. Then

$$\|A \sharp_\alpha B\| \leq \|A \diamondsuit_\alpha B\| \quad \text{for all } \alpha \in [0,1].$$

The log-majorization theorem due to Ando-Hiai: For each $\alpha \in (0,1]$

$$A^r \sharp_\alpha B^r \prec_{(\log)} (A \sharp_\alpha B)^r \qquad \text{for all } r \geq 1.$$

This implies

$$\left|\left|\left|(A^p \sharp_\alpha B^p)^{\frac{1}{p}}\right|\right|\right| \leq \left|\left|\left|(A^q \sharp_\alpha B^q)^{\frac{1}{q}}\right|\right|\right| \qquad \text{for } 0 < q < p.$$

Lie-Trotter formula is

$$A \diamondsuit_\alpha B = \lim_{q \to 0} (A^q \sharp_\alpha B^q)^{\frac{1}{q}} \qquad \text{for } \alpha \in (0,1]$$

Geometric mean of negative power

Ando-Hiai log-majorization of negative power

Let A, B be positive definite matrices and $\beta \in [-1, 0)$. Then

$$A^r \natural_\beta B^r \prec_{(\log)} (A \natural_\beta B)^r \qquad \text{for all } r \in (0,1] \qquad (*)$$

or equivalently

$$(A^q \natural_\beta B^q)^{\frac{1}{q}} \prec_{(\log)} (A^p \natural_\beta B^p)^{\frac{1}{p}} \quad \text{for all } 0 < q < p$$

Geometric mean of negative power

Proof

By the antisymmetric tensor power technique, in order to prove (*), it suffices to show that

$$\lambda_1(A^r \natural_\beta B^r) \le \lambda_1(A \natural_\beta B)^r \qquad \text{for all } 0 < r \le 1. \tag{1}$$

For this purpose we may prove that $A \natural_\beta B \le I$ implies $A^r \natural_\beta B^r \le I$, because both sides of (1) have the same order of homogeneity for A, B, so that we can multiply A, B by a positive constant.
First let us assume $\frac{1}{2} \le r \le 1$ and write $r = 1 - \varepsilon$ with $0 \le \varepsilon \le \frac{1}{2}$. Let $C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}$. Then $B^{-1} = A^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ and $A \natural_\beta B = A^{\frac{1}{2}} C^{-\beta} A^{\frac{1}{2}}$. If $A \natural_\beta B \le I$, then $C^{-\beta} \le A^{-1}$ so that $A \le C^\beta$ and $A^\varepsilon \le C^{\beta\varepsilon}$ for $0 \le \varepsilon \le \frac{1}{2}$ by Löwner-Heinz inequality.

Geometric mean of negative power

Proof

Since $-\beta \in (0,1]$ and $1-\varepsilon \in [\frac{1}{2},1]$, we now get

$$\begin{aligned} A^r \natural_\beta B^r &= A^{\frac{1-\varepsilon}{2}}(A^{\frac{\varepsilon-1}{2}}B^{1-\varepsilon}A^{\frac{\varepsilon-1}{2}})^\beta A^{\frac{1-\varepsilon}{2}} \\ &= A^{\frac{1-\varepsilon}{2}}(A^{\frac{1-\varepsilon}{2}}(B^{-1})^{1-\varepsilon}A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\ &= A^{\frac{1-\varepsilon}{2}}(A^{\frac{1-\varepsilon}{2}}(A^{-\frac{1}{2}}CA^{-\frac{1}{2}})^{1-\varepsilon}A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\ &= A^{\frac{1-\varepsilon}{2}}(A^{-\frac{\varepsilon}{2}}[A \sharp_{1-\varepsilon} C]A^{-\frac{\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\ &= A^{\frac{1}{2}-\varepsilon}[A^\varepsilon \sharp_{-\beta} (A \sharp_{1-\varepsilon} C)]A^{\frac{1}{2}-\varepsilon} \\ &\le A^{\frac{1}{2}-\varepsilon}[C^{\alpha\varepsilon} \sharp_{-\alpha} (C^\alpha \sharp_{1-\varepsilon} C)]A^{\frac{1}{2}-\varepsilon}, \end{aligned}$$

Geometric mean of negative power

Proof

using the joint monotonicity of matrix geometric means. Since a direct computation yields

$$C^{\beta\varepsilon} \sharp_{-\beta} (C^{\beta} \sharp_{1-\varepsilon} C) = C^{\beta(2\varepsilon-1)}$$

and by Löwner-Heinz inequality and $0 \le 1 - 2\varepsilon \le 1$, $C^{-\alpha} \le A^{-1}$ implies $C^{-\beta(1-2\varepsilon)} \le A^{-(1-2\varepsilon)}$ and thus we get

$$A^r \natural_{\beta} B^r \le A^{\frac{1}{2}-\varepsilon} C^{\beta(2\varepsilon-1)} A^{\frac{1}{2}-\varepsilon} \le A^{\frac{1}{2}-\varepsilon} A^{-1+2\varepsilon} A^{\frac{1}{2}-\varepsilon} = I.$$

Therefore (1) is proved in the case of $\frac{1}{2} \le r \le 1$.

Geometric mean of negative power

Proof

When $0 < r < \frac{1}{2}$, writing $r = 2^{-k}(1-\varepsilon)$ with $k \in \mathbb{N}$ and $0 \leq \varepsilon \leq \frac{1}{2}$, and repeating the argument above we have

$$\begin{aligned}\lambda_1(A^r \natural_\beta B^r) &\leq \lambda_1(A^{2^{-(k-1)}(1-\varepsilon)} \natural_\beta B^{2^{-(k-1)}(1-\varepsilon)})^{\frac{1}{2}} \\ &\vdots \\ &\leq \lambda_1(A^{1-\varepsilon} \natural_\beta B^{1-\varepsilon})^{2^{-k}} \\ &\leq \lambda_1(A \natural_\beta B)^r\end{aligned}$$

and so we have the Ando-Hiai log-majorization

$$A^r \natural_\beta B^r \prec_{(\log)} (A \natural_\beta B)^r \qquad \text{for all } r \in (0,1] \qquad (*)$$

Geometric mean of negative power

By the Ando-Hiai log-majorization of negative power,

$$\left|\left|\left|(A^q \natural_\beta \ B^q)^{\frac{1}{q}}\right|\right|\right| \leq \left|\left|\left|(A^p \natural_\beta \ B^p)^{\frac{1}{p}}\right|\right|\right| \qquad \text{for all } 0 < q < p$$

we have the following norm inequalities for geometric means of negative power:

Theorem 1

Let A and B be positive definite matrices. Then for every unitarily invariant norm

$$|||A \diamondsuit_\beta \ B||| \leq |||A \natural_\beta \ B||| \qquad \text{for all } \beta \in [-1, 0).$$

$$|||A \natural_\beta \ B||| \leq \left|\left|\left|A^{1-\beta} B^\beta\right|\right|\right| \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

Geometric mean of negative power

Proof of Theorem 1

By the Ando-Hiai log-majorization of negative power, it follows that

$$\left|\left|\left|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right|\right|\right| \le \left|\left|\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\right|\right| \qquad \text{for all } 0 < q < p$$

and as $q \to 0$ and $p = 1$ we have the desired inequality by the Lie-Trotter formula:

$$A \diamondsuit_\beta B = \lim_{q \to 0} (A^q \sharp_\beta B^q)^{\frac{1}{q}} \qquad \text{for } \beta \in [-1, 0)$$

and thus

$$|||A \diamondsuit_\beta B||| \le |||A \natural_\beta B||| \qquad \text{for all } \beta \in [-1, 0).$$

Geometric mean of negative power

Proof of Theorem 1

For the matrix norm $\|\cdot\|$, by the Araki-Cordes inequality $\|B^t A^t B^t\| \le \|(BAB)^t\|$ for all $t \in [0,1]$, we have for $-1 \le \beta \le -\frac{1}{2}$

$$\begin{aligned}
\|A \natural_\beta B\| &= \left\| A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\beta A^{\frac{1}{2}} \right\| \\
&= \left\| A^{\frac{1}{2}} (A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}})^{-\beta} A^{\frac{1}{2}} \right\| \\
&\le \left\| A^{-\frac{1}{2\beta}} A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}} A^{-\frac{1}{2\beta}} \right\|^{-\beta} \qquad \text{by } \frac{1}{2} \le -\beta \le 1 \\
&= \left\| A^{\frac{\beta-1}{2\beta}} B^{-1} A^{\frac{\beta-1}{2\beta}} \right\|^{-\beta} \\
&\le \left\| A^{1-\beta} B^{2\beta} A^{1-\beta} \right\|^{\frac{1}{2}} \qquad \text{for } \frac{1}{2} \le -\frac{1}{2\beta} \le 1 \\
&= \left\| (A^{1-\beta} B^{2\beta} A^{1-\beta})^{\frac{1}{2}} \right\|
\end{aligned}$$

Geometric mean of negative power

Proof of Theorem 1

and so we have

$$\lambda_1(A \natural_\beta B) \leq \lambda_1((A^{1-\beta}B^{2\beta}A^{1-\beta})^{\frac{1}{2}}) = \lambda_1(|B^\beta A^{1-\beta}|).$$

This implies

$$\prod_{i=1}^{k} \lambda_i(A \natural_\beta B) \leq \prod_{i=1}^{k} \lambda_i(|B^\beta A^{1-\beta}|) \qquad \text{for } k = 1, \ldots, n.$$

Hence we have the weak log majorization $A \natural_\beta B \prec_{w(\log)} \left|B^\beta A^{1-\beta}\right|$ and this implies

$$|||A \natural_\beta B||| \leq \left|\left|\left| \left|B^\beta A^{1-\beta}\right| \right|\right|\right| = \left|\left|\left| B^\beta A^{1-\beta} \right|\right|\right| = \left|\left|\left| A^{1-\beta}B^\beta \right|\right|\right|$$

for every unitarily invariant norm.

Geometric mean of negative power

Let A and B be positive definite matrices. Then

$$\|A \diamondsuit_\beta B\| \le \|A \natural_\beta B\| \qquad \text{for all } \beta \in [-1, 0).$$

$$\|A \natural_\beta B\| \le \left\| A^{1-\beta} B^\beta \right\| \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

Geometric mean of negative power

Let A and B be positive definite matrices. Then

$$\|A \diamondsuit_\beta B\| \le \|A \natural_\beta B\| \qquad \text{for all } \beta \in [-1, 0).$$

$$\|A \natural_\beta B\| \le \left\| A^{1-\beta} B^\beta \right\| \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

Remark

In Theorem 1, the inequality $\|A \natural_\beta B\| \le \left\| A^{1-\beta} B^\beta \right\|$ does not always hold for $-1/2 < \beta < 0$. In fact, if we put $\beta = -\frac{1}{3}$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then we have the matrix norm $\left\| A \natural_{-\frac{1}{3}} B \right\| = 3.385$ and $\left\| A^{\frac{4}{3}} B^{-\frac{1}{3}} \right\| = 3.375$, and so $\|A \natural_\beta B\| > \left\| A^{1-\beta} B^\beta \right\|$.

Geometric mean of negative power

Theorem 1

Let A and B be positive definite matrices. Then for every unitarily invariant norm

$$\|A \diamondsuit_\beta B\| \le \|A \natural_\beta B\| \le \left\| A^{1-\beta} B^{\beta} \right\| \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

Geometric mean of negative power

Theorem 1

Let A and B be positive definite matrices. Then for every unitarily invariant norm

$$\|A \diamondsuit_\beta B\| \le \|A \natural_\beta B\| \le \left\| A^{1-\beta} B^{\beta} \right\| \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

Conjecture

Let A and B be positive definite matrices. Then

$$w(A \diamondsuit_\beta B) \le w(A \natural_\beta B) \le w(A^{1-\beta} B^{\beta}) \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

counterexample

We would expect that the numerical radius inequality

$$w(A \natural_\beta B) \le w(A^{1-\beta}B^{\beta}) \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}].$$

However, the numerical radius $w(\cdot)$ is not unitarily invariant norm and unitarily similar. In fact, we have the following counterexample: We consider the case of $\beta = -\frac{1}{2}$. Put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$

Geometric mean of negative power

counterexample

and we have

$$A \natural_{-\frac{1}{2}} B = A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A^{\frac{3}{2}}B^{-\frac{1}{2}} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}.$$

Then we have $w(A^{\frac{3}{2}}B^{-\frac{1}{2}}) = \frac{1}{2}(\frac{2}{3} + \frac{\sqrt{5}}{3}) < w(A \natural_{-\frac{1}{2}} B) = \frac{\sqrt{5}}{3}$.
Hence

$$w(A \natural_{\beta} B) \leq w(A^{1-\beta}B^{\beta}) \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}]$$

does not always hold.

Geometric mean of negative power

counterexample

For the case of $\beta = -1$

$$A \natural_{-1} B = AB^{-1}A = \begin{pmatrix} \frac{5}{9} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A^2B^{-1} = \begin{pmatrix} \frac{5}{9} & -\frac{4}{9} \\ 0 & 0 \end{pmatrix}$$

Then we have $w(A^2B^{-1}) = \frac{1}{2}\left(\frac{5}{9} + \frac{\sqrt{41}}{9}\right) > \frac{5}{9} = w(A \natural_{-1} B)$.
Hence

$$w(A \natural_{\beta} B) \leq w(A^{1-\beta}B^{\beta}) \qquad \text{for } \beta = -1$$

hold.

Contents

Numerical radius inequality for geometric means

$$w(A \diamondsuit_\beta B) \le w(A \natural_\beta B) \qquad \text{for all } \beta \in [-1, 0)$$

and

$$w(A \diamondsuit_\beta B) \le w(A^{1-\beta} B^\beta) \qquad \text{for all } \beta \in \mathbb{R}$$

We show the following numerical radius inequalities releted to the geometric means of negative power:

Numerical radius inequality for geometric means

$$w(A \diamondsuit_\beta B) \le w(A \natural_\beta B) \qquad \text{for all } \beta \in [-1, 0)$$

and

$$w(A \diamondsuit_\beta B) \le w(A^{1-\beta} B^\beta) \qquad \text{for all } \beta \in \mathbb{R}$$

We show the following numerical radius inequalities releted to the geometric means of negative power:

Theorem 2

Let A and B be positive definite matrices. Then

$$w(A \natural_\beta B) \le w(A^{2(1-\beta)q} B^{2\beta q})^{\frac{1}{2q}} \qquad \text{for all } \beta \in [-1, -\tfrac{1}{2}] \text{ and } q \ge 1.$$

Numerical radius inequality for geometric means

Proof of Theorem 2

We recall the Araki-Cordes inequality for the matrix norm: If $A, B \geq 0$, then

$$\|B^p A^p B^p\| \leq \|BAB\|^p \qquad \text{for all } 0 < p \leq 1.$$

For $\beta \in [-1, -\frac{1}{2}]$, we have $\|A \natural_\beta B\| \leq \|A^q \natural_\beta B^q\|^{\frac{1}{q}}$ for all $q \geq 1$ and this implies

$$\begin{aligned}
w(A \natural_\beta B) &= \|A \natural_\beta B\| \leq \|A^q \natural_\beta B^q\|^{\frac{1}{q}} \quad \text{for all } q \geq 1 \\
&= \left\| A^{\frac{q}{2}} (A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^\beta A^{\frac{q}{2}} \right\|^{\frac{1}{q}} \\
&= \left\| A^{\frac{q}{2}} (A^{\frac{q}{2}} B^{-q} A^{\frac{q}{2}})^{-\beta} A^{\frac{q}{2}} \right\|^{\frac{1}{q}} \\
&\leq \left\| A^{\frac{-(1-\beta)q}{2\beta}} B^{-q} A^{\frac{-(1-\beta)q}{2\beta}} \right\|^{\frac{-\beta}{q}} \quad \text{for all } \frac{1}{2} \leq -\beta \leq 1
\end{aligned}$$

Numerical radius inequality for geometric means

Proof of Theorem 2

$$
\begin{aligned}
w(A \natural_\beta B) &\leq \left\| A^{\frac{-(1-\beta)q}{2\beta}} B^{-q} A^{\frac{-(1-\beta)q}{2\beta}} \right\|^{\frac{-\beta}{q}} \quad \text{for all } \tfrac{1}{2} \leq -\beta \leq 1 \\
&\leq \left\| A^{(1-\beta)q} B^{2\beta q} A^{(1-\beta)q} \right\|^{\frac{1}{2q}} \quad \text{for all } \tfrac{1}{2} \leq -\tfrac{1}{2\beta} \leq 1 \\
&= r(A^{(1-\beta)q} B^{2\beta q} A^{(1-\beta)q})^{\frac{1}{2q}} \\
&= r(A^{2(1-\beta)q} B^{2\beta q})^{\frac{1}{2q}} \\
&\leq w(A^{2(1-\beta)q} B^{2\beta q})^{\frac{1}{2q}}
\end{aligned}
$$

and so we have the desired inequality.

Corollary 3

Let A and B be positive definite matrices. Then

$$w(A \diamondsuit_\beta B) \le w(A^{\frac{1-\beta}{2}} B^\beta A^{\frac{1-\beta}{2}}) \le w(A \natural_\beta B) \le w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}}$$

for all $\beta \in [-1, -\frac{1}{2}]$.

(Proof) If we put $q = 1$ in Theorem 2, then we have this corollary.

Numerical radius inequality for geometric means

Finally, we consider the relation between $w(A^{1-\beta}B^{\beta})$ and $w(A^{2(1-\beta)}B^{2\beta})^{\frac{1}{2}}$. To show inequalities related to $w(A^{1-\beta}B^{\beta})$ for $\beta \in [-1, 0)$, we need the following Cordes type inequality related to the numerical radius:

Lemma 4

Let A and B be positive definite matrices. Then

$$w(AB) \leq w(A^{\frac{2}{p}}B^{\frac{2}{p}})^{\frac{p}{2}} \qquad \text{for all } p \in (0, 1]$$

Numerical radius inequality for geometric means

By Lemma 4, we have a series of the numerical radius inequalities related to the geometric means of negative power:

Theorem 5

Let A and B be positive definite matrices. Then

$$w(A \diamondsuit_\beta B) \le w(A^{\frac{1-\beta}{2}} B^\beta A^{\frac{1-\beta}{2}}) \le w(A^{1-\beta} B^\beta) \le w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}}$$

for all $\beta \in \mathbb{R}$.

Proof of Theorem 5

It follows that

$$w(A^{\frac{1-\beta}{2}}B^{\beta}A^{\frac{1-\beta}{2}}) = r(A^{\frac{1-\beta}{2}}B^{\beta}A^{\frac{1-\beta}{2}}) = r(A^{1-\beta}B^{\beta}) \le w(A^{1-\beta}B^{\beta})$$

and by Lemma 4

$$w(A^{1-\beta}B^{\beta}) \le w(A^{2(1-\beta)}B^{2\beta})^{\frac{1}{2}}$$

for all $\beta \in \mathbb{R}$.

Numerical radius inequality for geometric means

Comparison between $w(A \natural_\beta B)$ and $w(A^{1-\beta}B^\beta)$

Let A and B be positive definite matrices. Then

$$w(A \diamondsuit_\beta B) \le w(A^{\frac{1-\beta}{2}} B^\beta A^{\frac{1-\beta}{2}}) \le w(A \natural_\beta B) \le w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}}$$

for all $\beta \in [-1, -\frac{1}{2}]$.

$$w(A \diamondsuit_\beta B) \le w(A^{\frac{1-\beta}{2}} B^\beta A^{\frac{1-\beta}{2}}) \le w(A^{1-\beta} B^\beta) \le w(A^{2(1-\beta)} B^{2\beta})^{\frac{1}{2}}$$

for all $\beta \in \mathbb{R}$.

Numerical radius inequality for geometric means

Evaluation of $w(A \natural_\beta B)$ and $w(A^{1-\beta}B^\beta)$

Let A and B be positive definite matrices with $0 < m \le A, B \le M$ for some scalars $m < M$. Put $h = \frac{M}{m}$. If $-1 \le \beta \le -\frac{1}{2}$, then

$$K(h^2, -\beta)K(h, -2\beta)^{-\frac{1}{2}} w(A^{1-\beta}B^\beta) \le w(A \natural_\beta B) \le \frac{M+m}{2\sqrt{Mm}} w(A^{1-\beta}B^\beta)$$

where the generalized Kantorovich constant $K(h, p)$ is defined by

$$K(h, p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p$$

for any real number $p \in \mathbb{R}$.

Numerical radius inequality for geometric means

Evaluation of $w(A \natural_\beta B)$ and $w(A^{1-\beta}B^\beta)$

Let A and B be positive definite matrices with $0 < m \le A, B \le M$ for some scalars $m < M$. Put $h = \frac{M}{m}$. If $-\frac{1}{2} \le \beta < 0$, then

$$K(h^2, -\beta) w(A^{1-\beta}B^\beta) \le w(A \natural_\beta B) \le K(h, -\beta)^{-1} w(A^{1-\beta}B^\beta)$$

where $K(h, p)$ is the generalized Kantorovich constant.

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A. N. Kolmogorov

A. N. Kolmogorov's word

Finally, we present the following Professor A. N. Kolmogorov's word. He said in a lecture that

Behind every theorem lies an inequality.

Thank you very much for your attention !!