

# Further improvements of numerical radius inequalities

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- $\mathcal{B}(H)$ : the set of all bounded linear operator on Hilbert space  $H$ .
- For  $A \in \mathcal{B}(H)$ , operator norm  $\|A\| := \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|$ .
  - $\odot A = A^* \Rightarrow \|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle|$ .    $\odot \| |A| x \| = \|Ax\|, \forall A \in \mathcal{B}(H), \forall x \in H$ .
- For  $A \in \mathcal{B}(H)$ , numerical radius  $\omega(A) := \sup_{\|x\|=1} |\langle Ax, x \rangle|$ , where  $\|x\| := \sqrt{\langle x, x \rangle}$ .
  - $\odot A$  : normal  $\Rightarrow \|A\| = \omega(A)$ .    $\odot$  The property  $1/2\|A\| \leq \omega(A) \leq \|A\|$  is known.
- $I \subset (0, \infty)$ ,  $f: I \rightarrow (0, \infty)$ : **geometrically convex**

$$\xleftrightarrow{def} f(a^{1-v}b^v) \leq f^{1-v}(a)f^v(b), (a, b > 0, v \in [0, 1]). \cdots (\text{GC})$$

Ex.  $\exp$ ,  $\sinh$ ,  $\cosh$  on  $(0, \infty)$ .  $\tan$ ,  $\csc$ ,  $\sec$  on  $(0, \pi/2)$ .  $\arcsin$  on  $(0, 1]$ .  
 $-\log(1-x)$ ,  $(1+x)/(1-x)$  on  $(0, 1)$ .  $\sum_{n=0}^{\infty} c_n x^n$ , ( $c_n \geq 0$ ) on  $(0, R)$ .

**Advances** for  $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$ .

$$[\text{Kittaneh 2003}]: \omega(A) \leq \frac{1}{2} \left\| (A^* A)^{1/2} + (A A^*)^{1/2} \right\| \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{1/2} \right) \leq \|A\|.$$

$$[\text{El-Haddad\&Kittaneh 2007}]: \omega^r(A) \leq \frac{1}{2} \left\| |A|^{2rv} + |A^*|^{2r(1-v)} \right\|, (r \geq 1, v \in [0,1]).$$

$$[\text{Dragomir 2009}]: \omega^r(B^* A) \leq \frac{1}{2} \left\| (A^* A)^r + (B^* B)^r \right\|, (r \geq 1).$$

[\text{Shebrawi\&Albadawi 2009}]: For  $A, B, X \in \mathcal{B}(H)$ ,

$$\omega^r(A^* X B) \leq \frac{1}{2} \left\| (A^* |X^*|^{2v} A)^r + (B^* |X|^{2(1-v)} B)^r \right\|, (r \geq 1, v \in [0,1]).$$

[See e.g., Abu-Omar\&Kittaneh 2013 or Yamazaki 2007 and references therein.]

**Lemma 1**(Generalized mixed Schwarz inequality,Kittaneh 1988)

Let  $T \in \mathcal{B}(H)$  and  $x, y \in H$ . If  $f, g$  are continuous nonnegative functions on  $[0, \infty)$  satisfying  $f(t)g(t) = t$ , ( $t \geq 0$ ), then  $|\langle Tx, y \rangle| \leq \|f(|T|)x\| \cdot \|g(|T^*|)y\|$ .

**Lemma 2**(Jensen type inequality,Mond&Pecaric 1995)

If  $f$  is a convex function on a real interval  $J$  containing the spectrum of the self-adjoint operator  $A$ , then  $f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$ , ( $\forall x \in H, \|x\| = 1$ ).

**Lemma 3** (Aujla&Silva 2003) For positive operators  $A, B \in \mathcal{B}(H)$  and a nonnegative increasing convex function  $f$  on  $[0, \infty)$ . Then we have

$$\|f((1-\nu)A + \nu B)\| \leq \|(1-\nu)f(A) + \nu f(B)\|, \quad (\nu \in [0, 1]).$$

**Theorem 1** Let  $A, B \in \mathcal{B}(H)$ , and let  $f : (0, \infty) \rightarrow (0, \infty)$  be a continuous increasing convex function. Then,  $f(\omega(B^*A)) \leq \frac{1}{2} \|f(A^*A) + f(B^*B)\|$ .

$$\begin{aligned}
(\text{Proof}) \quad & f\left(\left|\langle B^*Ax, x \rangle\right|\right) = f\left(\left|\langle Ax, Bx \rangle\right|\right) \\
& \leq f(\|Ax\| \cdot \|Bx\|) \quad (\text{by Cauchy-Schwarz inequality}) \\
& = f\left(\sqrt{\langle Ax, Ax \rangle \langle Bx, Bx \rangle}\right) = f\left(\sqrt{\langle A^*Ax, x \rangle \langle B^*Bx, x \rangle}\right) \\
& \leq f\left(\frac{\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle}{2}\right) \quad (f : \text{increasing, AM-GM inequality}) \\
& \leq \frac{1}{2} f(\langle A^*Ax, x \rangle) + \frac{1}{2} f(\langle B^*Bx, x \rangle) \quad (f : \text{convex}) \\
& \leq \frac{1}{2} \left( \langle f(A^*A)x, x \rangle + \langle f(B^*B)x, x \rangle \right) \quad (\text{by Lemma 2})
\end{aligned}$$

By taking supremum over  $x \in H$  with  $\|x\|=1$  for

$$f\left(\left|\langle B^*Ax, x \rangle\right|\right) \leq \frac{1}{2} \left\langle \left(f(A^*A) + f(B^*B)\right)x, x \right\rangle$$

we obtain

$$\begin{aligned} f\left(\omega(B^*A)\right) &= f\left(\sup_{\|x\|=1} \left|\langle B^*Ax, x \rangle\right|\right) \\ &= \sup_{\|x\|=1} f\left(\left|\langle B^*Ax, x \rangle\right|\right) \quad (f: \text{continuous increasing}) \\ &\leq \frac{1}{2} \sup_{\|x\|=1} \left\langle \left(f(A^*A) + f(B^*B)\right)x, x \right\rangle \\ &= \frac{1}{2} \|f(A^*A) + f(B^*B)\|. \end{aligned}$$
□

∴  $f(t) = t^r$  ( $t > 0, r \geq 1$ ) satisfies the conditions in **Theorem 1**.

⇒ **Theorem 1** recovers [Dragomir 2009].

**Corollary 1** Let  $f$  as in **Theorem 1** and let  $A, B, X \in \mathcal{B}(H)$ . Then,

$$f\left(\omega(A^*XB)\right) \leq \frac{1}{2} \left\| f\left(A^*|X^*|^{2\nu} A\right) + f\left(B^*|X|^{2(1-\nu)} B\right) \right\|, \quad \nu \in [0,1]$$

(Proof) Let  $X = U|X|$  be the polar decomposition of  $X$ . Then,

$$f\left(\omega(A^*XB)\right) = f\left(\omega(A^*U|X|B)\right) = f\left(\omega\left(\left(|X|^{\nu} U^* A\right)^* \left(|X|^{1-\nu} B\right)\right)\right).$$

By substituting  $B = |X|^{\nu} U^* A$  and  $A = |X|^{1-\nu} B$  in **Theorem 1**, we obtain the desired inequality, noting that

$$X = U|X| \Rightarrow |X^*| = U|X|U^* \Rightarrow |X^*|^{2\nu} = U|X|^{2\nu} U^*.$$

□

•  $f(t) = t^r$  ( $t > 0$ ,  $r \geq 1$ ): **Corollary 1** recovers [Shebrawi&Albadawi 2009].

Another inequality for  $f(\omega(A^*XB))$

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•  $f$ : convex on  $(-\infty, \infty)$  and  $\alpha \leq 1 \Rightarrow f(\alpha t) \leq \alpha f(t) + (1 - \alpha)f(0) \cdots (*)$

( $\because$  decreasingness (resp., increasingness) of  $g(t)$  for  $t > 0$  (resp.,  $t < 0$ ))

•  $X$  : norm-contractive  $\xleftarrow[\text{def}]{\quad} \|X\| \leq 1$ ,  $X$  : norm-expansive  $\xleftarrow[\text{def}]{\quad} \|X\| \geq 1$ .

## Proposition 1

Under the same assumption as in **Theorem 1** with norm-contractive  $X$ ,

$$f(\omega(A^*XB)) \leq \frac{\|X\|}{2} \|f(A^*A) + f(B^*B)\| + (1 - \|X\|)f(0).$$

In particular, if  $f(0) = 0$ , then

$$f(\omega(A^*XB)) \leq \frac{\|X\|}{2} \|f(A^*A) + f(B^*B)\|.$$

**Corollary 2** Let  $A, B, X \in \mathcal{B}(H)$ . If  $X$  is norm-contractive, then

$$\omega^r(B^*XA) \leq \frac{\|X\|^r}{2} \left\| (A^*A)^r + (B^*B)^r \right\|, \quad (r \geq 1)$$

(Proof) A direct use of **Proposition 1** implies the weaker inequality:

$$\omega^r(B^*XA) \leq \frac{\|X\|}{2} \left\| (A^*A)^r + (B^*B)^r \right\|, \quad (r \geq 1).$$

However noting the proof of **Proposition 1** for  $f(t) = t^r$ , we have

$$f\left(\left|\langle B^*XAx, x \rangle\right|\right) \leq f(\|X\| \cdot \|Ax\| \cdot \|Bx\|) = f(\|X\|) f(\|Ax\| \cdot \|Bx\|).$$

Arguing as before implies the desired inequality. □

•  $X = I \Rightarrow$  **Corollary 2** recovers [Dragomir 2009].

**Theorem 2** Let  $A, B \in \mathcal{B}(H)$  and  $f : (0, \infty) \rightarrow (0, \infty)$  be a continuous increasing convex function. For  $\forall \alpha \in [0, 1]$ ,

$$f\left(\omega\left(\frac{A+B}{2}\right)\right) \leq \frac{1}{4} \left\| f\left(|A|^{2\alpha}\right) + f\left(|A^*|^{2(1-\alpha)}\right) + f\left(|B|^{2\alpha}\right) + f\left(|B^*|^{2(1-\alpha)}\right) \right\|.$$

- We omit the weighted case  $f\left(\omega((1-\nu)A + \nu B)\right)$  for  $\nu \in [0, 1]$ .
- $f(t) = t^r$ , ( $t > 0$ ,  $r \geq 1$ ) recovers [El-Haddad&Kittaneh2007]:

$$\omega^r(A+B) \leq 2^{r-2} \left\| |A|^{2\alpha r} + |A^*|^{2(1-\alpha)r} + |B|^{2\alpha r} + |B^*|^{2(1-\alpha)r} \right\|.$$

## Lemma 2'

If  $g$  is a **concave** function on a real interval  $J$  containing the spectrum of the self-adjoint operator  $A$ , then  $g(\langle Ax, x \rangle) \geq \langle g(A)x, x \rangle$ , ( $\forall x \in H, \|x\|=1$ ).

## Theorem 4

Let  $A \in \mathcal{B}(H)$ ,  $\alpha \in [0,1]$  and  $f : (0, \infty) \rightarrow (0, \infty)$  be a continuous increasing **geometrically convex** function. If in addition  $f$  is **convex**, then

$$f(\omega^2(A)) \leq \left\| \alpha f(|A|^2) + (1-\alpha) f(|A^*|^2) \right\|.$$

$\circledcirc f(t) = t^r, (t > 0, r \geq 1)$  recovers [El-Haddad&Kittaneh2007]:

$$\omega^{2r}(A) \leq \left\| \alpha |A|^{2r} + (1-\alpha) |A^*|^{2r} \right\|.$$

**Theorem 5** Let  $A, B, X \in \mathcal{B}(H)$ , and let  $f, g$  be continuous nonnegative functions on  $[0, \infty)$  satisfying  $f(t)g(t) = t$ , ( $\forall t > 0$ ). If  $h$  is a continuous nonnegative increasing convex function on  $[0, \infty)$ , then for any  $0 < \nu < 1$ ,

$$h\left(\omega^2(A^*XB)\right) \leq \left\| (1-\nu)h\left(\left(A^*g^2(|X^*|)A\right)^{1/(1-\nu)}\right) + \nu h\left(\left(B^*f^2(|X|)B\right)^{1/\nu}\right) \right\|.$$

In particular, ( $h(t) = t^{r/2}$ , ( $r \geq 2$ ) and  $\nu = 1/2$ )

$$\omega^r(A^*XB) \leq \frac{1}{2} \left\| \left(A^*g^2(|X^*|)A\right)^r + \left(B^*f^2(|X|)B\right)^r \right\|, \quad (r \geq 2),$$

which generalizes [Shebrawi&Albadawi 2009] for the case  $r \geq 2$ , since we take  $f(t) = t^\nu$  and  $g(t) = t^{1-\nu}$  for  $\nu \in [0, 1]$  as a special case.

**Lemma 4** Let  $a, b > 0$  and  $m, m', M, M' > 0$  satisfying

$0 < m' \leq \min\{a, b\} \leq m < M \leq \max\{a, b\} \leq M'$ . Then,

$$\frac{M+m}{2\sqrt{Mm}} \leq \frac{a+b}{2\sqrt{ab}} \leq \frac{M'+m'}{2\sqrt{M'm'}}.$$

(Proof) Consider  $f(x) := \frac{2\sqrt{x}}{1+x}$  on  $(1 \leq) \frac{M}{m} \leq x \leq \frac{M'}{m'}$ .

Since  $f'(x) = \frac{1-x}{\sqrt{x}(x+1)^2} \leq 0, (x \geq 1)$ , we get  $f\left(\frac{M'}{m'}\right) \leq f(x) \leq f\left(\frac{M}{m}\right)$ ,

which implies the result by a simple calculation. □

• We use 1st inequality to improve [Shebrawi&Albadawi 2009].

**Remark 1** For  $0 < m' \leq \min\{a, b\} \leq m < M \leq \max\{a, b\} \leq M'$ ,  
 we have the following **Zuo-Liao inequality**:

$$K^r(h)a \#_v b \leq a \nabla_v b \leq K^R(h')a \#_v b, \quad (r := \min\{\nu, 1 - \nu\}, \nu \in [0, 1])$$

where  $, h = \frac{M}{m}, h' = \frac{M'}{m'}, K(h) := \frac{(h+1)^2}{4h}$  : Kantorovich constant.

$\Rightarrow \frac{M+m}{2\sqrt{Mm}} \leq \frac{a+b}{2\sqrt{ab}}$  : special case of **Zuo inequality**, with  $\nu = \frac{1}{2}$ .

• We can replace the constant by other refined Young inequality.

**Ex.**  $S(h^r)a \#_v b \leq a \nabla_v b, S(h) := \frac{h^{1/(h-1)}}{e \log h^{1/(h-1)}}, (h \neq 1)$  : Specht ratio.

## Theorem 6

Let  $A, B, X \in \mathcal{B}(H)$ ,  $f, g$  be continuous nonnegative functions on  $[0, \infty)$  satisfying  $f(t)g(t) = t$ , ( $\forall t > 0$ ), and let  $h$  be a continuous nonnegative increasing convex function on  $[0, \infty)$ . If

$$0 < m' \leq B^* f^2(|X|)B \leq m < M \leq A^* g^2(|X^*|)A \leq M'$$

or

$$0 < m' \leq A^* g^2(|X^*|)A \leq m < M \leq B^* f^2(|X|)B \leq M',$$

then

$$h(\omega(A^* X B)) \leq \frac{\sqrt{Mm}}{M+m} \left\| h(B^* f^2(|X|)B) + h(A^* g^2(|X^*|)A) \right\|.$$

## **Remark 2 (Improvements for known results under some conditions)**

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Put  $f(t) = t^{1-\nu}$ ,  $g(t) = t^\nu$ ,  $h(t) = t^r$ , ( $t > 0, r \geq 1, \nu \in [0,1]$ ).

(i) Improvement of [Shebrawi&Albadawi 2009]:

$$\omega^r(A^*XB) \leq \frac{\sqrt{Mm}}{M+m} \left\| \left( A^* |X^*|^{2\nu} A \right)^r + \left( B^* |X|^{2(1-\nu)} B \right)^r \right\|, \quad (r \geq 1, \nu \in [0,1]).$$

(ii) Improvement of [El-Haddad&Kittaneh 2007]: ( $A = B = I$ )

$$\omega^r(X) \leq \frac{\sqrt{Mm}}{M+m} \left\| |X^*|^{2\nu r} + |X|^{2(1-\nu)r} \right\|, \quad (r \geq 1, \nu \in [0,1]).$$

(iii) Improvement of [Dragomir 2009]: ( $X = I$ )

$$\omega^r(B^*A) \leq \frac{\sqrt{Mm}}{M+m} \left\| |A|^{2r} + |B|^{2r} \right\|, \quad (r \geq 1).$$

## Alternative inequality by refined Young inequality (F., RACSAM2019)

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Let  $A, B > 0$  and  $m, m', M, M' > 0$  satisfying

$$0 < m' \leq A \leq m < M \leq B \leq M' \text{ or } 0 < m' \leq B \leq m < M \leq A \leq M'.$$

Then,  $1 \leq h \leq A^{-1/2}BA^{-1/2} \leq h'$  or  $\frac{1}{h'} \leq A^{-1/2}BA^{-1/2} \leq \frac{1}{h} \leq 1$ .

$$\exp_r \left( \frac{\nu(1-\nu)}{2} \left( 1 - \frac{\min\{1, t\}}{\max\{1, t\}} \right)^2 \right) \leq \frac{(1-\nu) + vt}{t^\nu}, \quad (\nu \in [0, 1], r \in [-1, 0])$$

$$\Rightarrow \exp_r \left( \frac{\nu(1-\nu)}{2} \left( 1 - \frac{1}{h'} \right)^2 \right) A \#_\nu B \leq A \nabla_\nu B, \quad \exp_r(x) := (1+rx)^{1/r}, \text{ if } 1+rx > 0.$$

•  $\exp_r x$ : decreasing in  $r \in [-1, 0]$ .  $\Rightarrow$  Tight lower bound is given in  $r = -1$ .

**Lemma 5** Let  $a, b > 0$  and  $m, m', M, M' > 0$  satisfying

$0 < \textcolor{blue}{m}' \leq \min\{a, b\} \leq m < M \leq \max\{a, b\} \leq \textcolor{blue}{M}'$ . Then,

$$\gamma \sqrt{ab} \leq \frac{a+b}{2}, \quad \gamma := \left(1 - \frac{1}{8} \left(1 - \frac{1}{\textcolor{blue}{h}'}\right)^2\right)^{-1} \geq 1, \quad \textcolor{blue}{h}' := \frac{\textcolor{blue}{M}'}{m'} \leq 1.$$

**Theorem 7** Under the same assumptions in **Theorem 6**, we have

$$h(\omega(A^* X B)) \leq \frac{1}{2\gamma} \left\| h(B^* f^2(|X|) B) + h(A^* g^2(|X^*|) A) \right\|.$$

- Improve some results as given in **Remark 2**. (Similar proof.)
- **Theorem 7** uses  $\textcolor{blue}{m}'$  and  $\textcolor{blue}{M}'$ , while **Theorem 6** does  $\textcolor{red}{m}$  and  $M$ .

## Lemma 6(An improvement of Lemma 3)

Under the same assumption with **Lemma 3**, we have

$$\left\| f((1-\nu)A + \nu B) \right\| \leq \left\| (1-\nu)f(A) + \nu f(B) \right\| - r_{\min} \cdot \mu_f,$$

where  $r_{\min} := \min \{ \nu, 1-\nu \}$  and

$$\mu_f := \inf_{\|x\|=1} \left\{ f(\langle Ax, x \rangle) + f(\langle Bx, x \rangle) - 2f\left(\left\langle \left( \frac{A+B}{2} \right)x, x \right\rangle\right) \right\}.$$

**(Proof)** The proof is a similar argument to the previous ones.

In addition, we show the estimate for the case  $\nu \in [0, 1/2]$ ,

since it is similar for the case  $\nu \in [1/2, 1]$ . For each unit  $x \in H$ ,

$$\begin{aligned}
& f \left( \left\langle ((1-\nu)A + \nu B)x, x \right\rangle \right) + r_{\min} \cdot \mu_f \\
&= f \left( (1-2\nu) \left\langle Ax, x \right\rangle + 2\nu \left\langle \left( \frac{A+B}{2} \right)x, x \right\rangle \right) + r_{\min} \cdot \mu_f \\
&\leq (1-2\nu) f \left( \left\langle Ax, x \right\rangle \right) + 2\nu f \left( \left\langle \left( \frac{A+B}{2} \right)x, x \right\rangle \right) + r_{\min} \cdot \mu_f \quad (f : \text{convex}) \\
&\leq (1-2\nu) f \left( \left\langle Ax, x \right\rangle \right) + 2\nu f \left( \left\langle \left( \frac{A+B}{2} \right)x, x \right\rangle \right) \quad (\text{definition of } \mu_f) \\
&\quad + \nu \left( f \left( \left\langle Ax, x \right\rangle \right) + f \left( \left\langle Bx, x \right\rangle \right) - 2 f \left( \left\langle \left( \frac{A+B}{2} \right)x, x \right\rangle \right) \right) \quad (r_{\min} = \nu \text{ for } \nu \in [0, 1/2]) \\
&= (1-\nu) f \left( \left\langle Ax, x \right\rangle \right) + \nu f \left( \left\langle Bx, x \right\rangle \right) \leq \left\langle ((1-\nu)f(A) + \nu f(B))x, x \right\rangle \quad (\textbf{Lemma 2}) \quad \square
\end{aligned}$$

**Corollary 3** Under the same assumption with **Theorem 5**, we have

$$h\left(\omega^2\left(A^*XB\right)\right) \\ \leq \left\|(1-v)h\left(\left(A^*g^2(|X^*|)A\right)^{1/(1-v)}\right) + vh\left(\left(B^*f^2(|X|)B\right)^{1/v}\right)\right\| - r_{\min} \cdot \mu_h,$$

where

$$\mu_h := \inf_{\|x\|=1} \left\{ h\left( \left\langle \left(A^*g^2(|X^*|)A\right)^{1/(1-v)} x, x \right\rangle \right) + h\left( \left\langle \left(B^*f^2(|X|)B\right)^{1/v} x, x \right\rangle \right) \right. \\ \left. - 2h\left( \left\langle \frac{\left( A^*g^2(|X^*|)A \right)^{1/(1-v)} + \left( B^*f^2(|X|)B \right)^{1/v}}{2} x, x \right\rangle \right) \right\}.$$

Recall  $f : \text{convex} \Rightarrow^{\exists} C(x)$  s.t.,  $f(y) - f(x) \geq C(x)(y - x)$ ,  $(\forall x, \forall y)$ .

$f : [0, \infty) \rightarrow \mathbb{R}$  : superquadratic  $\xleftarrow[\text{def}]{\forall} x \geq 0, \exists C(x)$  s.t.,

$$f(y) - f(x) \geq C(x)(y - x) + f(|y - x|), (\forall y \geq 0) \cdots (\diamond).$$

**Lemma 7** (S.Abramovich *et.al.*, 2004)

If  $f$  is non-negative superquadratic, then  $f$  is convex and increasing.

In addition,  $C(x) \geq 0$ .

**(Proof)** Since  $f(|y - x|) \geq 0$ ,  $f(y) - f(x) \geq C(x)(y - x)$ ,  $(\forall x, \forall y \geq 0)$ .

If  $y_1 < x < y_2$ , then  $\frac{f(x) - f(y_1)}{x - y_1} \leq C(x) \leq \frac{f(y_2) - f(x)}{y_2 - x}$  which implies

$$f(x) \leq \frac{y_2 - x}{y_2 - y_1} f(y_1) + \frac{x - y_1}{y_2 - y_1} f(y_2). \Rightarrow f : \text{convex}.$$

$f : \text{convex}$ ,  $f(0) = 0$  ( $x = y = 0 \rightarrow f(0) \leq 0$ , and  $f \geq 0$ ) and  $f(x) \geq 0$

$\Rightarrow f : \text{increasing}$ .

For  $x > 0$  and  $x > y \geq 0$ ,  $f(y) - f(x) - f(x-y) \geq C(x)(y-x)$ .

Thus we have,  $C(x) \geq \frac{f(x-y) - f(y) + f(x)}{x-y} \geq 0$ . □

•  $x^p$  : superquadratic for  $p \geq 2$ , and subquadratic for  $1 < p \leq 2$ .

Other examples of superquadratic:  $x^2 \log x, \sinh x$ ,

$f(x) := (x-a)^2$ , (if  $x > a$ ),  $0$ , (if  $0 \leq x \leq a$ ).

**Theorem 8** For  $A \in \mathcal{B}(H)$  and a non-negative superquadratic function  $f$ ,

$$f(\omega(A)) \leq \|f(|A|)\| - \inf_{\|x\|=1} \|f^{1/2}(|A| - \omega(A)) x\|^2.$$

$\odot f(t) = t^r$ , ( $r \geq 2$ ) is a superquadratic.

**Corollary 4** For  $A \in \mathcal{B}(H)$  and any  $r \geq 2$ ,

$$\omega^r(A) \leq \|A\|^r - \inf_{\|x\|=1} \left\| \left( |A| - \omega(A) \right)^{r/2} x \right\|^2.$$

In particular, (take  $r = 2$ )

$$\omega(A) \leq \sqrt{\|A\|^2 - \inf_{\|x\|=1} \| |A| - \omega(A) \| x \|^2} \leq \|A\|.$$

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## Geometrically convex functions and scalar inequalities

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**Lemma A** For geometrically convex function  $f$  on  $(0, \infty)$  and  $a, b > 0$ ,

$$f(a)^{1+v} f(b)^{-v} \leq f(a^{1+v} b^{-v}), \quad (\text{for any } v > 0 \text{ or } v < -1).$$

Equivalently,  $f(a) \#_v f(b) \leq f(a \#_v b)$ ,  $(v \notin [0, 1])$ .

**Theorem A1** For geometrically convex function  $f$  on  $(0, \infty)$  and  $a, b > 0$ ,

$$\left( \frac{f\sqrt{ab}}{\sqrt{f(a)f(b)}} \right)^{2R} \leq \frac{f(a^{1-v} b^v)}{f^{1-v}(a)f^v(b)} \leq \left( \frac{f\sqrt{ab}}{\sqrt{f(a)f(b)}} \right)^{2r}$$

where  $r := \min\{v, 1-v\}$ ,  $R := \max\{v, 1-v\}$  and  $v \in [0, 1]$ .

⊕ 1st inequality is reverse of (GC), 2nd one is refinement of (GC).<sup>26</sup>

## Theorem A2 ( $n$ -tuple version of Theorem A1)

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For geometrically convex function  $f$  on  $(0, \infty)$  and

$x_1, \dots, x_n > 0$ ,  $p_1, \dots, p_n \geq 0$  with  $\sum_{i=1}^n p_i = 1$ ,

$$\left( \frac{f\left(\left(\prod_{i=1}^n x_i\right)^{1/n}\right)}{\left(\prod_{i=1}^n f(x_i)\right)^{1/n}} \right)^{2R_n} \leq \frac{f\left(\prod_{i=1}^n x_i^{p_i}\right)}{\prod_{i=1}^n f^{p_i}(x_i)} \leq \left( \frac{f\left(\left(\prod_{i=1}^n x_i\right)^{1/n}\right)}{\left(\prod_{i=1}^n f(x_i)\right)^{1/n}} \right)^{2r_n}$$

where  $r_n := \min\{p_1, \dots, p_n\}$  and  $R_n := \max\{p_1, \dots, p_n\}$ .

**Note** From Lemma 4, we find with weight  $v \in [0,1]$  that if  $0 < a_n \leq a_{n-1} \leq \dots \leq a_1 < b_1 \leq \dots \leq b_{n-1} \leq b_n$ , then

$$1 \leq \frac{a_1 \nabla_v b_1}{a_1 \#_v b_1} \leq \dots \leq \frac{a_{n-1} \nabla_v b_{n-1}}{a_{n-1} \#_v b_{n-1}} \leq \frac{a_n \nabla_v b_n}{a_n \#_v b_n}.$$

Similarly we can get the following inequality

$$1 \leq \frac{a_1 \#_v b_1}{a_1 !_v b_1} \leq \dots \leq \frac{a_{n-1} \#_v b_{n-1}}{a_{n-1} !_v b_{n-1}} \leq \frac{a_n \#_v b_n}{a_n !_v b_n}.$$