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## Preservers of numerical radius on Lie products of self-adjoint operators

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15th Workshop on Numerical Ranges and Numerical Radii



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#### Outline

- Introduction
- General results
- Applications to numerical radius and numerical range,...



#### **1** Introduction

- We talk about the question of characterizing the general preservers of numerical radius and numerical range on Lie product of self-adjoint operators, but we first work on a more general setting.
- As we know, for some given subset *T* of matrices or operators, there are interesting results showing that φ : *T* → *T* will have nice structure if

$$F(\phi(A) \circ \phi(B)) = F(A \circ B) \ (A, B \in \mathcal{T})$$
(1.1)

for some suitable functional F and some product  $\circ$  of operators (see [CH], [CLS], [LPS], [Se] and the references therein).







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[CH] J.-L. Cui, J.-C. Hou, *Maps preserving functional values of operator products invariant*, Linear Algebra Appl. 428 (2008), 1649-1663.
[CLS] J.-T. Chan, C.-K. Li, N.-S. Sze, *Mappings on matrices: invariance of functional values of matrix products*, J. Austral. Math. Soc. 81 (2006) 165-184.
[LPS] C.-K. Li, E. Poon, N.-S. Sze, *Preservers for norms of Lie product*, Oper. Matrices 3 (2009), 187-203.
[Se] P. Šemrl, *Maps on matrix spaces*, Linear Algebra Appl. 413 (2006), 364-

393.

For example, the paper [Dob] gives a characterization of the maps φ satisfying (1.1) when the operation ∘ is the Jordan semi-triple product (i.e., A ∘ B = ABA) and T is a subset of the algebra M<sub>n</sub> of all n × n complex matrices which contains all idempotent and nilpotent rank one matrices, and F : M<sub>n</sub> → [0,∞) is a unitary similarity invariant function satisfying the following properties:

(i)  $F(UAU^*) = F(A)$  for every  $A \in \mathcal{M}_n$  and unitary  $U \in \mathcal{M}_n$ ;

(ii)  $F(\lambda A) = |\lambda| F(A)$  for every  $\lambda \in \mathbb{C}$  and  $A \in \mathcal{M}_n$ ; (iii)  $F(A) = 0 \Leftrightarrow A = 0$ .







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• This general result is then applied to characterize the maps that preserve the norm, the maps that preserve the Schatten *p*-norm and the maps preserving the numerical radius of Jordan semi-triple products of matrices.

[Dob] G. Dobovišek, B. Kuzma, G. Lešnjak, C.-K. Li and T. Petek, *Mappings that preserve pairs of operators with zero triple Jordan product*, Linear Algebra Appl. 426 (2007), 255-279.

- However, the pseudo spectral radius does not have the properties (ii) and (iii).
- More generally, the paper [CLP] introduces a radial unitary similarity function F : M<sub>n</sub> → [0, ∞), that is, F has the property

(I)  $F(\lambda UAU^*) = F(A)$  for every unimodular  $\lambda \in \mathbb{C}$ ,  $A \in \mathcal{M}_n$  and unitary  $U \in \mathcal{M}_n$ .

In addition,

(II)  $F(A) = 0 \Leftrightarrow A = 0$ ;

(III) For every rank one nilpotent  $R \in \mathcal{M}_n$ , the map  $t \mapsto F(tR)$  is strictly increasing on  $[0, \infty)$ .

[CLP] J.-L. Cui, C.-K. Li, Y.-T. Poon, *Preservers of unitary similarity functions on Lie products of matrices*, Linear Algebra Appl. 498 (2016), 160-180.





- With the continuous functional F satisfying (I)-(III), a characterization of the maps on  $\mathcal{M}_n$  satisfying (1.1) when the operation  $\circ$  is the Lie product  $(A \circ B = [A, B] = AB BA)$  is obtained.
- Then, as applications, general forms of the maps that preserve the pseudo spectral radius and the maps that preserve the pseudo spectrum are presented.
- Let B(H) stand for the algebra of all bounded linear operator acting on a Hilbert space H with dim H ≥ 3. In the recent paper [Ben] the authors characterized all maps φ on B(H) satisfying (1.1) when the operation ∘ is the Jordan semi-triple product, and F : B(H) → [0,∞) satisfies modified conditions (I)-(III) where M<sub>n</sub> is replaced by B(H).

[Ben] M. Bendaoud, A. Benyouness, M. Sarih, S. Sekkat, *Preservers of radial unitary similarity functions on products of operators*, Linear Algebra Appl, 542 (2018), 484-500.





For a complex Hilbert space H with dim H ≥ 3, denote by B<sub>s</sub>(H) the set of all bounded self-adjoint operators on H.

[XuHou] Qingsen Xu, Jinchuan Hou, Preservers of radial unitary similarity functions on Jordan semi-triple products of self-adjoint operators, Bull. Malays. Math. Sci. Soc., https://doi.org/10.1007/s40840-018-06104.

• In [XuHou] we consider the following radial unitary similarity invariant functionals

$$F: \mathcal{B}(H) \to [d, \infty]$$

with  $d \ge 0$  which have the following properties  $P_1$ - $P_4$ :



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- (P<sub>1</sub>)  $F(\lambda UAU^*) = F(A)$  for any scalar  $\lambda$  with  $|\lambda| = 1$ ,  $A \in \mathcal{B}(H)$  and unitary operator  $U \in \mathcal{B}(H)$ ;
- (P<sub>2</sub>)  $F(A) = d \Leftrightarrow A = 0$  for  $A \in \mathcal{B}(H)$ ;
- (P<sub>3</sub>) For every rank one projection P ∈ B(H), the map t → F(tP) is strictly increasing on [0, ∞);
- (P<sub>4</sub>)  $F(A) < \infty$  for every rank one operator A.
- Note that allowing F may take the value +∞ at some operators is more reasonable especially in the infinite dimensional case. This enables us to cover also the Schatten p-norm || · ||p.





• We got the following general result for Jordan semi-triple product:

Theorem XH. Let H be a complex Hilbert space with  $\dim H \geq 3$  and let  $F : \mathcal{B}(H) \rightarrow [d, \infty]$  be a functional satisfying  $P_1 - P_4$ . If  $\phi : \mathcal{B}_s(H) \rightarrow \mathcal{B}_s(H)$  is a surjective map such that

 $F(\phi(A)\phi(B)\phi(A)) = F(ABA) \quad \forall A, B \in \mathcal{B}_s(H),$ 

then there exists a unitary or conjugate unitary operator U on H and a functional  $h : \mathcal{B}_s(H) \to \{-1, 1\}$  such that

 $\phi(T) = h(T)UTU^* \quad \forall T \in \mathcal{B}_s(H).$ 

• Then it was applied to characterize the general preservers of numerical radius, pseudo-spectral radius, Schatten *p*-norm.





#### **2** General results for Lie product

- Now let us turn to the problem how to characterize the maps that preserve the value of some radial unitary similarity function on Lie products of self-adjoint operators.
- Here we consider the following functional  $F : \mathcal{B}(H) \rightarrow [d, \infty]$  with  $d \ge 0$  satisfying the following properties  $P'_1 P'_4$ :
- $(P'_1) F(\lambda UAU^*) = F(A)$  for all scalar  $\lambda$  with  $|\lambda| = 1$  and any  $A, U \in \mathcal{B}(H)$  with U unitary operator;
- $(\mathbf{P}'_2)$  For any  $A \in \mathcal{B}(H)$ ,  $F(A) = d \Leftrightarrow A = 0$ ;
- (P'\_3) For every nonzero  $R \in \mathcal{B}(H)$  with rank $R \leq 2$  and  $\operatorname{Tr}(R) = 0$ , the map  $t \mapsto F(tR)$  on  $[0, \infty)$  is strictly increasing;
- $(P'_4) F(A) < \infty$  for every operator A of rank  $\leq 2$ .



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• Denote by  $\mathcal{D}_s(H)$  the set of all diagonalizable self-adjoint operators in  $\mathcal{B}(H)$ .

Lemma 2.1. Let  $F : i\mathcal{B}_s(H) \to [d, \infty]$  with  $d \ge 0$ be a functional satisfying the properties  $P'_1 \cdot P'_4$ . For any  $A \in \mathcal{D}_s(H)$  and  $B \in \mathcal{B}_s(H)$ , the following statements equivalent:

(1) For any  $C \in \mathcal{B}_s(H)$ ,  $F(AC - CA) = d \Leftrightarrow F(BC - CB) = d$ .

(2) For any unit vector  $x \in H$ ,  $F(Ax \otimes x - x \otimes xA) = d \Leftrightarrow F(Bx \otimes x - x \otimes xB) = d$ .

(3) There exist  $\mu \in \{-1, 1\}$  and  $\nu \in \mathbb{R}$  such that  $B = \mu A + \nu I$ .



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Lemma 2.2. Let  $F : i\mathcal{B}(H) \to [d, \infty]$  be a functional satisfying  $P'_1 \cdot P'_4$  with  $d \ge 0$ . For an orthonormal basis  $\{e_k\}_{k \in \Lambda}$ of H and every  $x = \sum_{k \in \Lambda} \xi_k e_k \in H$ , define  $\overline{x} = Jx =$  $\sum_{k \in \Lambda} \overline{\xi}_k e_k$ . Then, for any  $B \in \mathcal{B}_s(H)$ , we have

 $F([B,\overline{x}\otimes\overline{x}]) = F([B^{t},x\otimes x]).$ 

• The following is our general result.

Theorem 2.3. Let H be a complex separable Hilbert Space of dimension  $\geq 3$ , and let  $F : i\mathcal{B}_s(H) \to [d, \infty]$ with  $d \geq 0$  be a functional satisfying the properties  $P'_1 \cdot P'_4$ . Suppose  $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$  is a surjective map satisfying

$$F(\phi(A)\phi(B) - \phi(B)\phi(A)) = F(AB - BA) \quad \forall A, B \in \mathcal{B}_s(H).$$

Then there exist a unitary or conjugate unitary operator U on H, a sign function  $h : \mathcal{D}_s(H) \to \{1, -1\}$ and a functional  $g : \mathcal{D}_s(H) \to \mathbb{R}$  such that  $\phi(A) = h(A)UAU^* + g(A)I$  for all  $A \in \mathcal{D}_s(H)$ .



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• Denote by  $\mathbf{H}_n$  the space of all  $n \times n$  Hermitian matrices. Particularly, we have

**Corollary 2.4.** Let  $F : i\mathbf{H}_n \to [d, \infty]$  with  $d \ge 0$  and  $n \ge 3$  be a functional satisfying the properties  $P'_1 - P'_4$ . Suppose  $\phi : \mathbf{H}_n \to \mathbf{H}_n$  is a surjective map satisfying

 $F(\phi(A)\phi(B)-\phi(B)\phi(A))=F(AB-BA) \ \, \forall A,B\in \mathcal{B}_s(H).$ 

Then there exist a unitary matrix  $U \in \mathcal{M}_n$ , a sign function  $h : \mathbf{H}_n \to \{1, -1\}$  and a functional  $g : \mathbf{H}_n \to \mathbb{R}$  such that (1)  $\phi(T) = h(T)UTU^* + g(T)I$  for all  $T \in \mathbf{H}_n$ ; or

(2)  $\phi(T) = h(T)UT^{t}U^{*} + g(T)I$  for all  $T \in \mathbf{H}_{n}$ .

# **3** Applications: numerical radius, numerical range, Schatten *p*-norm, pseudo spectral radius, pseudo spectrum

- Now let us consider some applications of the general result Theorem 2.3.
- Let  $\varepsilon$  be a small positive number. For an operator  $A \in \mathcal{B}(H)$ , recall that the  $\varepsilon$ -pseudo spectrum of A is the set

$$\sigma_{\varepsilon}(A) = \{ \mu \in \mathbb{C} : \| (A - \mu I)^{-1} \| \ge \varepsilon^{-1} \}$$

with the convention that  $||(A - \mu I)^{-1}|| = \infty$  if  $\mu \in \sigma(A)$ , and the  $\varepsilon$ -pseudo spectral radius of A is

 $r_{\varepsilon} = \max\{|\mu| : \mu \in \sigma_{\varepsilon}(A)\}.$ 



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- Obviously, for any ε > 0, we have σ(A) ⊆ σε(A) and r(A) ≤ rε(A), where σ(A) and r(A) are respectively the spectrum and the spectral radius of A.
- The pseudo spectrum and the pseudo spectral radius are useful in studying the stability of operators under perturbations. The preserver problems concerning the pseudo spectrum and the pseudo spectral radius have been studied by many authors.

**Theorem 3.1.** Let  $p \ge 1$ ,  $\varepsilon > 0$  and H be a complex separable Hilbert space of dimension at least 3. Assume that  $\Phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$  is a surjective map. Then, the following conditions are equivalent.

(1)  $w([\Phi(A), \Phi(B)]) = w([A, B])$  for all  $A, B \in \mathcal{B}_s(H)$ . (2) For some  $1 \le p < \infty$ ,  $\|[\Phi(A), \Phi(B)]\|_p = \|[A, B]\|_p$  for all  $A, B \in \mathcal{B}_s(H)$ . (3)  $r_{\varepsilon}([\Phi(A), \Phi(B)]) = r_{\varepsilon}([A, B])$  for all  $A, B \in \mathcal{B}_s(H)$ . (4) There exist a unitary or conjugate unitary operator U on H, a sign function  $h : \mathcal{B}_s(H) \to \{1, -1\}$  and a functional  $g : \mathcal{B}_s(H) \to \mathbb{R}$  such that either

$$\Phi(T) = h(T)UTU^* + g(T)I$$

for all  $T \in \mathcal{B}_s(H)$ .



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#### • Set

 $\mathfrak{S} = \{A \in \mathcal{B}_s(H) : \sigma(A) \text{ has at most two points} \}.$ 

• The following result gives a characterization of maps preserving numerical range, preserving the pseudo spectrum of Lie product of self-adjoint operators. Theorem 3.2. Let  $\varepsilon > 0$ , H be a complex Hilbert space with dim  $H \ge 3$  and  $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$  a surjective map. Then the following statements are equivalent. (1)  $\phi$  satisfies

 $W(\phi(A)\phi(B) - \phi(B)\phi(A)) = W(AB - BA) \qquad \forall A, B \in \mathcal{B}_s(H).$ 

(2)  $\phi$  satisfies

 $\sigma_{\varepsilon}(\phi(A)\phi(B)-\phi(B)\phi(A)) = \sigma_{\varepsilon}(AB-BA) \qquad \forall A, B \in \mathcal{B}_s(H).$ 

(3) There exist a unitary operator on H, a functional g:  $\mathcal{B}_s(H) \to \mathbb{R}$ , a scalar  $\eta \in \{-1, 1\}$  and a subset  $\Delta \subseteq \mathfrak{S}$ , such that

$$\phi(T) = \begin{cases} \eta UTU^* + g(T)I & \text{if } T \notin \Delta; \\ -\eta UTU^* + g(T)I & \text{if } T \in \Delta. \end{cases}$$







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#### • Question:

- We do not know if the similar results in this talk can still be established when dim  $H = \infty$  and  $\mathcal{B}_s(H)$  is replaced by  $\mathcal{B}(H)$ .
- Particularly, is the following conjecture true?

**Conjecture.** Let *H* be an infinite-dimensional complex Hilbert Space, and let  $F : \mathcal{B}(H) \to [d, \infty]$  with  $d \ge 0$ be a functional satisfying the properties  $P'_1 \cdot P'_4$ . Suppose  $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$  is a surjective map satisfying

 $F(\phi(A)\phi(B)-\phi(B)\phi(A))=F(AB-BA) \ \, \forall A,B\in \mathcal{B}(H).$ 

Then there exist a unitary or conjugate unitary operator U on H, a function  $h : \mathcal{D}(H) \to \mathbb{T}$  and a functional  $g : \mathcal{D}(H) \to \mathbb{C}$  such that  $\phi(A) = h(A)UA^{\#}U^* + g(A)I$  for all  $A \in \mathcal{D}(H)$ , where  $A^{\#} = A$  or  $A^*$ .

•  $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  and  $\mathcal{D}(H)$  is the set of all diagonalizable operators in  $\mathcal{B}(H)$ .







### Thank You for Your Attention!

