The diameter and width of the numerical range

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Let A be an $n \times n$ complex matrix. The numerical range of A is defined and denoted by

$$W(A) = \{x^* A x; x \in \mathbb{C}^n, \|x\| = 1\}.$$

The homogeneous polynomial:

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

and $\Re(A) = (A + A^*)/2, \Im(A) = (A - A^*)/(2i)$.

Gau, Helton, Spitkovsky, Vinnikov, Zyczkowski, ...

The diameter $\mathrm{diam}(W(A))$ of $W(A)$ is defined to be the largest distance of two parallel lines tangent to its boundary.

The width $\mathrm{width}(W(A))$ the smallest distance of two parallel lines tangent to its boundary.

The boundary curve of $W(A)$ is called *a curve of constant width* if $\mathrm{diam}(W(A)) = \mathrm{width}(W(A))$.

Our work

- Provide an algorithm for computing the diameter and width of the numerical range.
- Formulate the diameter and width of the numerical range of some unitary bordering matrices.
- Determine the condition for the boundary of the numerical range of certain Toeplitz matrices to be a curve of constant width.

An algorithm for computing the diameter and width

Let A be an $n \times n$ matrix. For $0 \leq \theta \leq 2\pi$, we consider the Cartesian decomposition

$$e^{-i\theta}A = \Re(e^{-i\theta}A) + i\Im(e^{-i\theta}A).$$

Denote $H_A(\theta) = \Re(e^{-i\theta}A)$, and its eigenvalues

$$\lambda_1(\theta) \geq \lambda_2(\theta) \geq \ldots \geq \lambda_n(\theta).$$

$$e^{-i\theta}A = (\cos\theta - i\sin\theta)(\Re(A) + i\Im(A)),$$

it follows that

$$H_A(\theta) = \cos\theta\,\Re(A) + \sin\theta\,\Im(A).$$

Hence, the characteristic polynomial of $H_A(\theta)$ is exactly equal to

$$F_A(t, -\cos\theta, -\sin\theta) = \det(tI - \cos\theta\,\Re(A) - \sin\theta\,\Im(A)).$$

Theorem 1 Let A be an $n \times n$ matrix. Then

$$\mathrm{diam}(W(A)) = \max\{\lambda_1(\theta) - \lambda_n(\theta) : 0 \leq \theta \leq 2\pi\} \tag{2.1}$$

and

$$\mathrm{width}(W(A)) = \min\{\lambda_1(\theta) - \lambda_n(\theta) : 0 \leq \theta \leq 2\pi\}, \tag{2.2}$$

where $\lambda_1(\theta) \geq \lambda_2(\theta) \geq \cdots \geq \lambda_n(\theta)$ are eigenvalues of $H_A(\theta)$.

The resultant of two polynomials $f(Y), g(Y)$ due to Sylvester:

Let

$$f(Y) = a_m Y^m + a_{m-1} Y^{m-1} + \cdots + a_1 Y + a_0$$
$$g(Y) = b_n Y^n + b_{n-1} Y^{n-1} + \cdots + b_1 Y + b_0$$

be polynomials in Y with non-zero leading coefficients a_m, b_n, and the coefficients a_j, b_k are functions in other variables.

The resultant of $f(Y)$ and $g(Y)$ is defined as the determinant of the $(n+m)\times(n+m)$-matrix:

$$R(f,g) = \begin{pmatrix} a_m & a_{m-1} & a_{m-2} & \dots & 0 & 0 & 0 \\ 0 & a_m & a_{m-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \dots & a_2 & a_1 & a_0 \\ b_n & b_{n-1} & b_{n-2} & \dots & 0 & 0 & 0 \\ 0 & b_n & b_{n-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \dots & b_2 & b_1 & b_0 \end{pmatrix}.$$

It is well known f and g have a common zero iff $\det(R(f,g)) = 0$.

Obviously, the function $\lambda_1(\theta) - \lambda_n(\theta)$ is a zero of the polynomial

$$\prod_{1 \le j \ne k \le n} (Z - (\lambda_j(\theta) - \lambda_k(\theta))).$$

This means that once we obtained the above polynomial, the diameter $\mathrm{diam}(W(A))$, according to Theorem 1, can be derived.

Consider the polynomial

$$p(t) = t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n = \prod_{1 \le i \le n} (t - \lambda_i).$$

Assume $R(Z)$ is the resultant of $p(Y + Z)$ and $p(Y)$ with respect to Y. Then $R(Z)/Z^n$ is the required polynomial:

$$\prod_{1 \le j \ne k \le n} (Z - (\lambda_j - \lambda_k)).$$

Accordingly, let $R(Z, -\cos\theta, -\sin\theta)$ be the resultant of the polynomials $F_A(Y, -\cos\theta, -\sin\theta)$ and $F_A(Y+Z, -\cos\theta, -\sin\theta)$ with respect to Y. Then $R(Z, -\cos\theta, -\sin\theta)/Z^n$ is

$$K(Z, -\cos\theta, -\sin\theta) = \prod_{1 \le j \ne k \le n} (Z - (\lambda_j(\theta) - \lambda_k(\theta))).$$

Theorem 2 Let A be an $n \times n$ matrix. Let $R(Z, x, y)$ be the resultant of $F_A(Y, x, y)$ and $F_A(Y+Z, x, y)$ with respect to Y, and $K_A(Z, x, y) = R(Z, x, y)/Z^n$. Then

$$\mathrm{diam}(W(A)) = \max\{Z \in \mathbb{R} : K_A(Z, -\cos\theta, -\sin\theta) = 0,\ 0 \le \theta \le 2\pi\},$$

$$\mathrm{width}(W(A)) = \min_{0 \le \theta \le 2\pi} \max\{Z \in \mathbb{R} : K_A(Z, -\cos\theta, -\sin\theta) = 0\}.$$

Changing the variable $s = \tan(\theta/2)$,

$$\mathrm{diam}(W(A)) = \sup\left\{Z \in \mathbb{R} : K_A(Z, -\frac{1-s^2}{1+s^2}, -\frac{2s}{1+s^2}) = 0, -\infty < s < \infty\right\},$$

$$\mathrm{width}(W(A)) = \inf\left\{\sup\{Z \in \mathbb{R} : K_A(Z, -\frac{1-s^2}{1+s^2}, -\frac{2s}{1+s^2}) = 0\},\right.$$

$$\left. -\infty < s < \infty\right\}.$$

Example 1

Consider the matrix

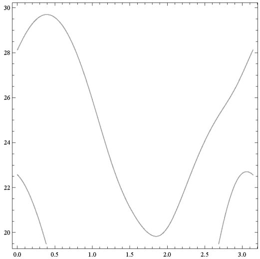
$$A = \begin{pmatrix} 23 & 1+i & 6-5i \\ 4+5i & 6i & 7 \\ 2+i & -3+5i & -9i \end{pmatrix}.$$

Then the polynomial $K_A(Z,x,y)$ becomes

$$K_A(Z,x,y) = 16Z^6 - 16(1331x^2 + 378xy + 645y^2)Z^4 + 4(1331x^2 + 378xy + 645y^2)^2 Z^2$$
$$-198638261x^6 - 785100590x^5y - 1971769127x^4y^2 - 503512476x^3y^3 - 238036399x^2y^4$$
$$-88713198xy^5 - 77494717y^6.$$

The resultant of $K_0(Z,s)$ and its derivative $K_0'(Z,s)$ with respect to s contains the following polynomial factor :

$$L(Z) = 136878346901135360 0Z^{24} + \cdots$$



$K(Z, -\cos\theta, -\sin\theta) = 0.$

The maximal real root of $L(Z) = 0$ is approximately 29.6976 which is $\mathrm{diam}(W(A))$. The minimal real root is approximately 19.8313 which is $\mathrm{width}(W(A))$.

Formulate diameter and width of unitary bordering matrices

An $n \times n$ complex matrix A is called a unitary bordering matrix
if A is a contraction, that is, $\langle A^*A\xi, \xi\rangle \leq \langle \xi, \xi\rangle$ for $\xi \in \mathbb{C}^n$,
$\mathrm{rank}(I_n - A^*A) = 1$
and the modulus of any eigenvalue of A is strictly less than 1.

Gau-Wu(1998), Mirman(1998):
The entries of a standard form of a unitary bordering matrix $A = (a_{ij})$ in the upper triangular form, up to unitary equivalence, are determined by its eigenvalues $a_1, a_2, \ldots, a_n$ in the following way

$$a_{ij} = \begin{cases} a_i & \text{if } i = j \\ \left(\prod_{k=i+1}^{j-1}(-\overline{a_k})\right)\sqrt{(1-|a_i|^2)(1-|a_j|^2)} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}.$$

For simplicity, we consider an $n \times n$ unitary bordering matrix with eigenvalues

$$\{a \exp(i\frac{2k\pi}{n}) : k = 0, 1, 2, \ldots, n-1\}$$

for $0 < a < 1$.
The standard form of such an $n \times n$ unitary bordering matrix A is denoted by $A_n(a)$.

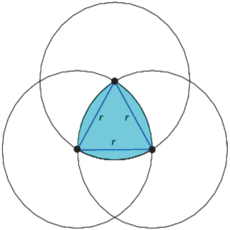
By the resultant formulae of Theorem 2, we compute that

Theorem 3 Let $A_n(a)$ be the standard unitary bordering matrix as defined above. Then,

1. For $n = 2$, $\mathrm{diam}(W(A_2(a))) = 1 + a^2$ and $\mathrm{width}(W(A_2(a))) = 1 - a^2$,
2. For $n = 3$, $\mathrm{diam}(W(A_3(a))) = (2 + a^6)^{1/2}$ and $\mathrm{width}(W(A_3(a))) = (2 + (a^6/4))^{1/2}$,
3. For $n = 4$, $\mathrm{diam}(W(A_4(a))) = \frac{1}{\sqrt{2}}\sqrt{a^8 + 3 + \sqrt{a^{16} + 2a^8 + 8a^4 + 5}}$ and $\mathrm{width}(W(A_4(a))) = \frac{1}{\sqrt{2}}\sqrt{a^8 + 3 + \sqrt{a^{16} + 2a^8 - 8a^4 + 5}}$.

Curve of constant width for certain Toeplitz matrices

Reuleaux triangle:



Reuleaux triangle:curve of constant width

Conjecture:

If C is a curve of constant width and it is the boundary of the numerical range of a matrix, then C is a circle or a single point.

Denote $T(\beta_1, \ldots, \beta_{n-1})$ the $n \times n$ nilpotent Toeplitz matrix:

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{n-2} & \beta_{n-1} \\ 0 & 0 & \beta_1 & \beta_2 & \ldots & \beta_{n-3} & \beta_{n-2} \\ 0 & 0 & 0 & \beta_1 & \ldots & \beta_{n-4} & \beta_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

If $n = 2m$ is even, we denote

$$A(\beta_1, \beta_2, \ldots, \beta_{m-1}, \beta_m) = T(\beta_1, \beta_2, \ldots, \beta_{m-1}, \beta_m, \overline{\beta_{m-1}}, \ldots, \overline{\beta_2}, \overline{\beta_1}).$$

If $n = 2m - 1$ is odd,

$$A(\beta_1, \beta_2, \ldots, \beta_{m-1}) = T(\beta_1, \beta_2, \ldots, \beta_{m-1}, \overline{\beta_{m-1}}, \ldots, \overline{\beta_2}, \overline{\beta_1}).$$

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Theorem 4 (1) If $n = 2m - 1$ and $B = A(\beta_1, \ldots, \beta_{m-1})$, then the eigenvalues $\rho_k(\theta)$ of the Hermitian matrix $H_B(\theta)$ are given by

$$\rho_k(\theta) = (-1)^k \Re\Big(\sum_{j=1}^{m-1} \beta_{m-j} \exp(-i(\frac{(2j-1)\theta}{n} + \frac{(2j-1)k\pi}{n}))\Big).$$

(2) If $n = 2m$ and $B = A(\beta_1, \ldots, \beta_m)$, then the eigenvalues $\rho_k(\theta)$ of the Hermitian matrix $H_B(\theta)$ are given by

$$\rho_k(\theta) = (-1)^k \frac{\Re(\beta_m)}{2} + (-1)^k \Re\Big(\sum_{j=1}^{m-1} \beta_{m-j} \exp(-i(\frac{2j\theta}{n} + \frac{2jk\pi}{n}))\Big),$$

$k = 0, 1, 2, \ldots, n-1$.

Theorem 5 Let B be an $n \times n$ nilpotent Toeplitz matrix $A(\beta_1, \ldots, \beta_{m-1})$ (resp. $A(\beta_1, \ldots, \beta_m)$) for $n = 2m - 1$ (resp. $n = 2m$). If $\partial W(B)$ is a curve of constant width then $W(B) = \{0\}$ if $n = 2m - 1$, and $W(B) = \{z \in \mathbb{C} : |z| \leq r\}$ for some $r \geq 0$ if $n = 2m$.

Theorem 6 Let B be the nilpotent Toeplitz matrix $A(\beta, \ldots, \beta_{m-1})$ or $A(\beta, \ldots, \beta_m)$ for $n = 2m - 1$ or $n = 2m$. If $p(\theta) = \rho_0(\theta)$, $k = 0$ in Theorem 4, then the boundary generating curve of $W(B)$ is given by

$$\Re(z(\theta)) = p(\theta)\cos\theta - p'(\theta)\sin\theta,$$

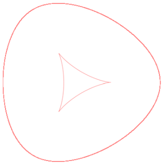
$$\Im(z(\theta)) = p(\theta)\sin\theta + p'(\theta)\cos\theta,$$

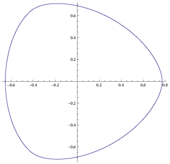
$0 \leq \theta \leq 2\pi$.

We compare the boundary generating curves of $W(A(3, 3/5))$ and the curve of constant width by Rabinowitz curve.

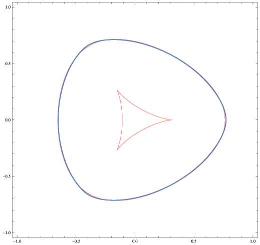
Boundary generating curve of $W(A(3, 3/5))$: $p(\theta)$ in Theorem 6.

Rabinowitz curve: $p(\theta) = \frac{3}{25}\cos^2(\frac{3\theta}{2}) + \frac{3261}{5000}$.





$W(A(3, 3/5))$ Rabinowitz curve.



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Thank you