Numerical ranges, operator systems, and quantum channels

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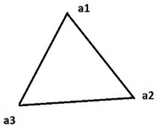

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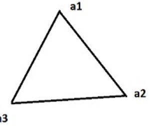

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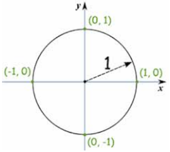
- If $T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $W(T) = \{z \in \mathbb{C} : |z| \leq 1\}$, the unit disk centered at the origin.



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- Note that $W(A) = W(I \otimes A)$. There are results showing that $W(T) \subseteq W(A)$ ensures that $T \in B(H)$ has a dilation of the form $I \otimes A$.

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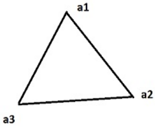

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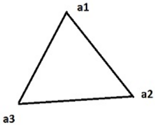

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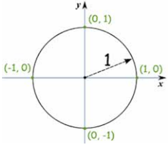

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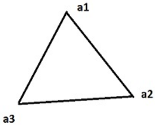

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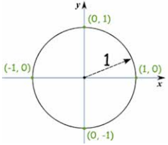

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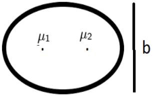

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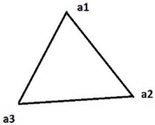

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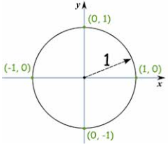

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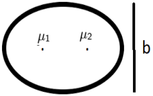
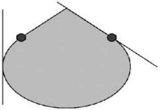


- (Choi and Li, 2001) $A \in M_3$ has a reducing eigenvalue, i.e., A is unitarily similar to $[\alpha] \oplus A_0$ with $A_0 \in M_2$.



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$$T \text{ has a dilation of the form } I \otimes A.$$

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The study of OMAX is related to the study of quantum channels such as entanglement breaking channels; see Paulsen et al. (2017).

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- The joint numerical range of self-adjoint operators $T_1, \ldots, T_m \in B(H)$ is

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- If $T_j = H_j + iG_j$ for self-adjoint H_j, K_j for $j = 1, \ldots, m$, then

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- Let $\mathcal{S}$ have a basis $\{I_n, A_1, \ldots, A_m\}$ such that $A_1, \ldots, A_m$ are Hermitian. Then $\mathcal{S}$ is an OMAX if and only if $(T_1, \ldots, T_m) \in B(H)^m$ has a joint dilation of the form $(I \otimes A_1, \ldots, I \otimes A_m)$ whenever
$$W(T_1, \ldots, T_m) \subseteq \operatorname{conv} W(A_1, \ldots, A_m).$$

Extension of Mirman's result

- If $W(T_1, T_2, T_3)$ lies inside in a simplex in $\mathbb{R}^3$ with vertices:

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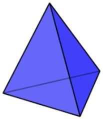
- Note that one can choose any $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ as long as

$$W(T_1, T_2, T_3) \subseteq \mathrm{conv}\,\{v_1, v_2, v_3, v_4\}.$$



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That is, there is a partial isometry X such that

$$X^*(I_N \otimes A_j)X = T_j, \quad j = 1, \ldots, m.$$

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Corollary

An operator system $\mathcal{S} = \mathrm{span}\,\{I, A_1, A_2, A_3\} \subseteq M_3$ is an OMAX if $W(A_1, A_2, A_3)$ is a non-degenerate ice-cream cone,

i.e., the convex hull of a point and an elliptical disk with non-empty interior.

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- There is a basis $\{I_n, A_1, \ldots, A_m\}$ for $\mathcal{S}(\Phi)$ consisting of Hermitian matrices.
- Properties of $\mathcal{S}(\Phi)$, and hence Φ can be studied via certain generalized numerical ranges of $(A_1, \ldots, A_m)$.

- (Paulsen et al, 2017; Li, Poon and Watrous, 2019+) If $\mathcal{S} \subseteq M_n$ is an operator system, then there is a quantum operation $\Psi : M_n \to M_k$ such that $\mathcal{S} = \mathcal{S}(\Psi)$. If $\dim \mathcal{S} \le \ell^2$, then we can have $k < n$.

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Numerical Ranges and Operator Systems

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- One may get "better" quantum error correction code if there is an $n \times k$ matrix X such that $X^*X = I_k$ and

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with positive integers $p_1, \dots, p_k$, diagonal / square matrices $B_{1j}, \dots, B_{rj}$.

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Thank you for your attention!